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A CALCULATION OF HIGHER ORDER QCD CORRECTIONS TO 2- AND 3-JET
OBSERVABLES IN e^+e^- -ANNIHILATION ON AND OFF THE Z^0 INCLUDING BEAM
POLARIZATION EFFECTS

by

G. Schuler

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**A CALCULATION OF HIGHER ORDER QCD
CORRECTIONS TO 2- AND 3-JET OBSERVABLES
IN e^+e^- -ANNIHILATION ON AND OFF THE Z^0
INCLUDING BEAM POLARIZATION EFFECTS**

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Abstract

We present $O(\alpha_s^2)$ - corrections to the following 3-jet observables in e^+e^- - annihilation: (i) the complete set of structure functions that describe the space orientation of 3-jet events, (ii) charge asymmetries in $e^+e^- \rightarrow \gamma, Z^0 \rightarrow 3\text{-jets}$, and (iii) polarizations of final state partons. We describe many calculational aspects of the problem which should be of help for further higher order QCD calculations. In particular we discuss how to deal with anomalies which arise within dimensional regularization of parity violating quantities in the presence of IR/M singularities.

Wer an sich erfahren hat, was ein reichhaltiger
Gedanke heissen will, er sei nun aus uns selbst
entsprungen, oder von andern mitgeteilt und
eingepft, wird gestehen, was dadurch fuer
eine leidenschaftliche Bewegung in unserm
Geiste hervorgebracht werde, wie wir uns
begeistert fuehlen, indem wir alles dasjenige in
Gesamtheit vorausahnen, was in der Folge
sich mehr und mehr entwickeln, wozu das
Entwickelte weiter fuehren soll.

Goethe

INDEX OF NOTATIONS

variables of 2-parton final state (i=1,2)

$$r_i \quad 2\text{-parton process momenta: } r_i^2 = 0$$

variables of 3-parton final state (i,j=1,2,3)

$$q_i \quad 3\text{-parton process momenta: } q_i^2 = 0$$

$$t_{ij} \ (i \neq j) \quad \text{invariant mass squared} \quad t_{ij} = (q_i + q_j)^2 \equiv 2q_i q_j$$

$$z_{ij} \ (i \neq j) \quad \text{scaled invariant mass squared} \quad z_{ij} = t_{ij}/q^2$$

$$x_i \quad \text{scaled parton energies} \quad x_i = \frac{2q_i q}{q^2} = \frac{2q_{i0}}{\sqrt{q^2}} = 1 - z_{jk}$$

variables of 4-parton final state (i,j,k=1,2,3,4)

$$p_i \quad 4\text{-parton process momenta: } p_i^2 = 0$$

$$s_{ij} \ (i \neq j) \quad \text{invariant mass squared} \quad s_{ij} = (p_i + p_j)^2$$

$$y_{ij} \ (i \neq j) \quad \text{scaled invariant mass squared} \quad y_{ij} = s_{ij}/q^2$$

$$s_{ijk} \ (i \neq j \neq k \neq i) \quad s_{ijk} = s_{ij} + s_{jk} + s_{ik}$$

$$q \quad \text{momentum of the virtual } \gamma \text{ or } Z;$$

$$q = r_1 + r_2 = q_1 + q_2 + q_3 = p_1 + p_2 + p_3 + p_4$$

hadron tensors for 2-jet calculations (X = p.c., p.v.)

$$B_{\mu\nu}^{(2)X}(r_1, r_2) \equiv H_{\mu\nu}^X[2\text{-partons}; \alpha_s^0] \quad \text{Born hadron tensor}$$

$$H_{\mu\nu}^{(2)X}(r_1, r_2; Y) \equiv H_{\mu\nu}^X[2\text{-jet}; \alpha_s]$$

$$= H_{\mu\nu}^{(2)X}(r_1, r_2)_{\text{virt}} + H_{\mu\nu}^{(2)X}(r_1, r_2; Y)_{\text{real}}$$

$$H_{\mu\nu}^{(2)X}(r_1, r_2)_{\text{virt}} \equiv H_{\mu\nu}^X[2\text{-partons}; \alpha_s]$$

$$H_{\mu\nu}^{(2)X}(r_1, r_2; Y)_{\text{real}} \equiv \int dR \ H_{\mu\nu}^X[3\text{-partons}; \alpha_s]$$

unresolved pairs

hadron tensors for 3-jet calculations (X = p.c., p.v.)

$$B_{\mu\nu}^{(3)X}(q_1, q_2, q_3) \equiv H_{\mu\nu}^X[3\text{-partons}; \alpha_s] \quad \text{Born hadron tensor}$$

$$\begin{aligned} H_{\mu\nu}^{(3)X}(q_1, q_2, q_3; Y) &\equiv H_{\mu\nu}^X[3\text{-jet}; \alpha_s^2] \\ &= H_{\mu\nu}^{(3)X}(q_1, q_2, q_3)_{\text{virt}} + H_{\mu\nu}^{(3)X}(q_1, q_2, q_3; Y)_{\text{real}} \end{aligned}$$

$$H_{\mu\nu}^{(3)X}(q_1, q_2, q_3)_{\text{virt}} \equiv H_{\mu\nu}^X[3\text{-partons}; \alpha_s^2]$$

$$H_{\mu\nu}^{(3)X}(q_1, q_2, q_3; Y)_{\text{real}} \equiv \int dR \quad H_{\mu\nu}^X[4\text{-partons}; \alpha_s^2]_{\text{unresolved pairs}}$$

$$B_{\mu\nu}^{(4)X}(p_1, p_2, p_3, p_4) \equiv H_{\mu\nu}^X[4\text{-partons}; \alpha_s^2]$$

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1. INTRODUCTION

Quantum Chromodynamics (QCD) can be tested unambiguously at short distances using perturbation theory without having to solve the confinement and bound state problems, since it is an asymptotically free theory of the hadron interactions. The study of hadron jets in e^+e^- annihilation provides one of the cleanest tests of perturbative QCD. True, the non-perturbative contributions have either to be modelled or extracted from the data. Yet, the successes of perturbation theory in e^+e^- annihilation are convincing and the QCD tests have reached a precision level which is both quantitative and impressive /8,16/. To quote a few examples, one could cite the prediction of the total hadronic cross section /94/, the existence of quark jets first proven in QCD by Sterman and Weinberg /102/, the prediction /40/ and discovery /15/ of gluon jets in e^+e^- annihilation, tests of the quark and gluon spins /41/, calculations of 3-jet production to $O(\alpha_s^2)$ /42,105,46,74,75,72/ and measurements of the effective coupling constant $\alpha_s(Q^2)$ /107,16,32,3,106/.

In this paper we perform $O(\alpha_s)$ calculations of 2-jet events and $O(\alpha_s^2)$ calculations of 3-jet events in e^+e^- annihilation. The cm energy of the present e^+e^- machines is high enough to warrant the application of perturbative QCD. Much experimental /18,17,2,20,10,4/ and theoretical work /6,7,42,45,46,50,53,54,64,73,105/ has been expended on the elucidation of the role that $O(\alpha_s^2)$ corrections play in the description of the 3-jet events in e^+e^- annihilation. These have been found to be nonnegligible and thus effect the predictions of 3-jet events.

The first $O(\alpha_s^2)$ calculations were done for the trace of the hadronic tensor, i.e. the $O(\alpha_s^2)$ corrections to space- direction averaged 3-jet events /6,7,42,45,46,50,51,54,74,75,105/. These were used as input for the determination of the strong coupling constant α_s from the 3-jet cross section calculated to $O(\alpha_s^2)$. With more data it will be desirable to check also on the $O(\alpha_s^2)$ corrections to oriented 3-jet events, or vice versa, it will be desirable to be able to generate $O(\alpha_s^2)$ oriented 3-jet events via Monte Carlo /73,64/¹. Further, one would like to have available $O(\alpha_s^2)$ corrections to polarization type observables of the final state partons.

In the energy range around $\sqrt{s} \approx 30\text{GeV}$ and above, weak interactions become more and more important in e^+e^- annihilation. Electromagnetic amplitudes ($\approx e^2/s \approx 10^{-1}/s$) and weak amplitudes ($\approx G_F \approx 10^{-5}/\text{GeV}^2$) are of comparable magnitude in the 60-70GeV region, giving rise to substantial electromagnetic- weak interference effects. In this paper we will discuss 3-jet events in e^+e^- annihilation throughout the cm energy range $\sqrt{s} \geq 30\text{GeV}$ where both electromagnetic and weak production mechanisms contribute (Fig. 1)

$$(1) \quad e^+e^- \rightarrow \gamma, Z^0 \rightarrow q\bar{q}$$

both on and off the Z^0 resonance. We will deal with the spin kinematics of process (1) within the context of electro- weak models. Due to the Z^0 coupling we encounter contributions which violate parity (P) and charge conjugation (C) invariance, but conserve CP-invariance.

¹For $O(\alpha_s)$ results see /63,68,76,77,85,95,96/.

The general space orientation of 3-jet events in e^+e^- annihilation is described by 5 parity conserving (p.c.) and 4 parity violating (p.v.) structure functions, or equivalently by 5 p.c. and 4 p.v. helicity cross sections. In massless QCD one of the p.c. and two of the p.v. structure functions (helicity cross sections) can be shown to vanish identically to $O(\alpha_s^2)$ /62,65/. In this work we will evaluate the remaining 4 p.c. and 2 p.v. structure functions to $O(\alpha_s^2)$.

Up to now only the angular averaged 3-jet cross section has been measured. The statistical accuracy of the e^+e^- data is not yet sufficient for the determination of the individual helicity cross sections. Such measurements will be possible in the near future with higher statistics coming from PETRA and PEP, TRISTAN, SLC and LEP. Since these structure functions are calculable in QCD such measurements provide crucial tests of the detailed structure of QCD. Also one obtains independent measurements of the quark-gluon coupling constant. For latter measurement it is essential to know the structure functions at least up to $O(\alpha_s^2)$ since the coupling can be defined uniquely through renormalization only in this order or higher order in perturbation theory.

The p.v. asymmetries that are induced by the p.v. hadronic structure are measurable in the present high energy machines and are sizeable in the energy range available to the next generation e^+e^- machines (TRISTAN, SLC, LEP). Off the Z^0 these charge asymmetries result from the usual electro weak γ - Z^0 interference effects, whereas a measurement on the Z^0 requires longitudinally polarized Z^0 's due to the smallness of the leptonic vector current contribution proportional to $(1-4\sin^2\theta_W)$. Longitudinally

polarized Z^0 's are planned to be produced at the SLC using longitudinally polarized electrons and/or positrons for the annihilation process. The measurements of the p.v. asymmetries do, however, require quark flavour tagging, which will reduce the available data sample.

In this paper we will mainly be concerned with the p.c. and p.v. structure functions on and off the Z^0 as they are measured in e^+e^- annihilation. We have therefore worked out the details of the spin kinematics including beam polarization effects for this process in section 2. Up to now Z 's (and similarly W 's) have been produced only in $p\bar{p}$ collisions at the collider at CERN. Note that the p.c. and p.v. contributions to the decays $Z \rightarrow q\bar{q}G$ and $W \rightarrow q'\bar{q}G$ are described by the same 4 p.c. and 2 p.v. structure functions calculated in this paper. Thus our $O(\alpha_s^2)$ results apply also to the decay of hadronically produced Z 's and W 's although we do not work out the details of the spin kinematics of these decay processes. Lowest order calculations of the polarization of hadronically produced W 's and Z 's (at high and low p_T) have been given in /27,29,12,39,47/.

From $O(\alpha_s)$ calculations /86,87/ it is known that the gluons emitted in e^+e^- annihilation at high energies have a remarkably high degree of linear polarization. Measurements of the gluon polarization e.g. through angular correlations in the decay of isoscalar vector mesons or through gluon jet oblateness could constitute sensitive tests of QCD. In particular, this would be a sensitive, direct test of the vector property of the gluon. It is crucial to know whether higher order QCD corrections substantially modify the predicted polarization of the gluon. Therefore we will

procedure must be done order by order in the coupling constant α_s . In this way, G.Sterman and S.Weinberg/¹QCD/ derived the cross section for e^+e^- annihilation into 2 jets to lowest order in α_s .

Possible choices of IR/M cut-off parameters are combined energy-angle cuts or an invariant mass cut-off⁴. With these cut-off parameters N-parton final states are divided into $n \leq N$ jet substates: Using e.g. an invariant mass cut-off y , two (or more) partons are viewed as irresolvable if their invariant squared mass satisfies $(\sum p_i)^2 \leq y \cdot s$ where $\sqrt{s}$ is the cms energy. A final n-jet cross section of a given order $O(\alpha_s^k)$ is finally arrived at by summing up all $O(\alpha_s^k)$ virtual (loop) and real (tree) contributions integrated over the unresolved regions of phase space.

In section 3 and 4 we discuss further subtleties in connection with the appropriate definition of 2-jet and 3-jet cross sections and the domain of integration corresponding to the unresolved regions of phase space. Unfortunately, there is no obviously "correct" way to define resolvable 3-jet or 4-jet events, respectively /52/. Different assignement of certain regions of phase space as either being 2-jet or 3-jet (3-jet or 4-jet) lead to different results for the corresponding jet cross sections.

We choose to work with massless quarks (and gluons) and use dimensional regularization /56/ to control UV (see below) and IR/M infinities. We use as cut-off parameter an invariant mass cut-off

⁴ Since the final results depend on the cut-off parameters, these must be chosen in accordance with experimental requirements.

present new $O(\alpha_s^2)$ calculations to both the gluon's linear and circular polarization. Furthermore we will discuss the influence of $O(\alpha_s^2)$ corrections to the polarization of quark and antiquark allowing for further tests of the quark spin and QCD.

On the theoretical side higher order QCD calculations are confronted with the problem that parton cross sections in general carry infinities. The general framework of the occurrence of such infrared (IR) and mass (M) singularities and their cancellations is given by the KLN theorem /61/. The origin of the divergences can be traced back to the fact that a system containing zero mass particles² has a high degree of degeneracy. However, averaging the relevant transition probability over an appropriate ensemble of degenerate states removes the infinities.

The application of the KLN theorem to jet cross sections is known as jet dressing /42,53,54/. We remove IR divergences by replacing a single quark state by a state including its soft gluons. On the other hand, we get rid of the mass or collinear divergences³ by replacing any fast moving quark or gluon in a reaction by a "jet" of particles, i.e. an ensemble of massless partons which all have their 3-momentum lying in a small cone in the momentum space. Of course, in perturbation theory this

² The gluons are massless and the quarks can be treated as massless at the energy scale considered in this paper.

³ Mass or collinear divergencies arise in the limit of vanishing quark masses.

γ and perform our IR/M integrations up to $O(\gamma^0)$, i.e. we restrict ourself to constant terms and to terms linear in $\ln\gamma$. One can always choose a small enough γ to make this a safe approximation. If necessary, corrections for higher cut-off values can be added numerically.

The calculation of p.v. quantities involving P-odd fermion traces requires a careful discussion of how to deal with γ_5 and the totally antisymmetric ϵ -tensor $\epsilon(\alpha\beta\gamma\delta)$ within dimensional regularization (see e.g. /83/). The n-dimensional P-odd Dirac traces generate $O((n-4)^m; m>0)$ anomalous contributions, which in turn lead to finite anomalous terms when multiplied with UV and/or IR/M divergent integrals.

Spurious UV anomalies can be and must be cancelled by taking the appropriate renormalization scheme /23,79,104/. In the IR/M case one encounters axial, gauge and charge conjugation anomalies /66/. These cannot be removed by renormalization. For the 2-jet and 3-jet cross sections discussed in this paper we demonstrate that the IR/M charge conjugation anomaly vanishes after IR/M integration and that the finite axial and gauge anomalies are spurious in the sense that they cancel among the real and virtual contributions to the p.v. structure functions as do the IR/M singular terms /64/.

We use the γ_5 dimensional regularization scheme of Breitenlohner and Maison (BM) /25/ which, to our knowledge, is the only n-dimensional γ_5 -scheme free of internal inconsistencies. The BM scheme was originally developed for the regularization of UV singularities in the presence of γ_5 . We found that the BM scheme is also well suited for the treatment of IR singularities on

in the presence of γ_5 . For the purpose of our calculation the BM scheme can be fully implemented by observing the following two simple rules: i) do not commute γ_5 and ii) the trace

$-\frac{1}{4}\text{Tr}\gamma_5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta$ equals the conventional ϵ -tensor $\epsilon(\alpha\beta\gamma\delta)$ in 4 dimensions if the indices $\alpha,\beta,\gamma,\delta$ are "4-dimensional" but equals zero otherwise. A more complete account of the BM γ_5 -scheme together with a discussion of possible other versions of dimensional regularization is given in section 5 and Appendix A and B.

Since the presence of γ_5 in the p.v. case brings in some novel features compared to the p.c. case we will discuss them carefully. We will treat these features in detail since we feel that the experience gained from this first explicit calculation of a higher order QCD correction to a p.v. cross section will be quite valuable for many other higher order perturbative calculations that have to be done for the interpretation of p.v. experiments at the high energy machines to be completed in the following years.

In calculating virtual (loop) corrections to the cross sections we encounter two (three) types of singularities. Besides the IR/M divergences the loop amplitudes carry in general ultraviolet (UV) infinities. Before we can obtain a finite answer these singularities have to be regularized during the intermediate steps of the calculation and the UV poles have to be removed via renormalization. We will show in section 6 how the virtual corrections can be calculated in a very elegant way regularizing the UV and IR divergences simultaneously by n-dimensional regularization. It is even possible to treat all singularities on

the same footing maintaining nevertheless the ability to perform the renormalization, i.e. to subtract the UV poles.

In section 7 we present numerical results for the $O(\alpha_s^2)$ corrected quantities calculated in this paper. We will present our results in terms of single variable thrust distributions. Furthermore, we present plots for the charge asymmetries and the final state polarization distributions, and add some comments on the usefulness of beam polarization.

Finally, in section 8 we provide the summary and present our conclusions. In particular we emphasize the occurrence of some systematic features in our final results and speculate about their general validity.

2. e^+e^- ANNIHILATION INTO PARTONS

We consider electron-positron annihilation in the 1γ - and/or $1 Z^0$ -exchange approximation. The virtual photon γ and the neutral intermediate boson Z^0 couple to the final state through a quark-antiquark pair of flavour f , which then turns into a n -parton final state via hard gluon bremsstrahlung and pair production (Fig. 2). The cross section depends most generally on the beam polarization, the angular correlations between the initial and final states, the flavour of the produced quark-antiquark pair, the couplings of the gauge bosons to the fermions and the dynamics of the final state /63,95,72,48,69,76,59,35,14,44,96/. The task of the chapter is to discuss these various dependences and to decompose the cross section into spin-kinematical and leptonic and hadronic dynamical parts.

2.1 Specification of the electro-weak model dependence

Following closely the notation of /96/ we define the electromagnetic and weak couplings by the following interaction Lagrangean densities

$$(2.1.1) \quad L_{e.m.} = -e \left[Q_e \bar{\psi}_e \gamma^\mu \psi_e + Q_f \bar{\psi}_f \gamma^\mu \psi_f \right] A_\mu$$

$$L_{n.c.} = -e m_Z \sqrt{g} \left[\bar{\psi}_e \gamma^\mu (v_e - a_e \gamma_5) \psi_e + \bar{\psi}_f \gamma^\mu (v_f - a_f \gamma_5) \psi_f \right] Z_\mu$$

where the Dirac fields $\psi_e, \bar{\psi}_e$ and the vector fields A_μ, Z_μ describe, respectively, the electron e^- , the quark with flavour f , the photon γ and the neutral intermediate boson Z^0 of mass m_Z ; Q_e and Q_f give the electric charge of e^- and quark of flavour f in

units of $e = \sqrt{4\pi\alpha}$, where $\alpha = \frac{1}{137}$ is the fine structure constant. The parameters $v(v_f)$ and $a(a_f)$ determine the strength of the vector and axial vector couplings of the Z^0 at the electron (quark) vertex in terms of the dimensionless coupling constant $e m_Z \sqrt{g}$, where g is related to the Fermi weak coupling constant G_F by

$$(2.1.2) \quad g = \frac{G_F}{8\pi\alpha\sqrt{2}} \approx 4.49 \times 10^{-5} \text{ GeV}^{-2}.$$

For example in the Standard model one has for

(2.1.3)

(i) charged leptons

$$Q_e = -1, \quad v = -1 + 4 \sin^2 \theta_W, \quad a = -1$$

(ii) u, c, t, ... quarks

$$Q_f = +\frac{2}{3}, \quad v_f = +1 - \frac{8}{3} \sin^2 \theta_W, \quad a_f = +1$$

(iii) d, s, b, ... quarks

$$Q_f = -\frac{1}{3}, \quad v_f = -1 + \frac{4}{3} \sin^2 \theta_W, \quad a_f = -1$$

Here θ_W is the Weinberg angle related to g by

$$(2.1.4) \quad m_Z \sqrt{g} = \frac{1}{2 \sin 2\theta_W}$$

The Feynman amplitude for the process

$$(2.1.5) \quad e^-(p_-) + e^+(p_+) \rightarrow \gamma, Z^0(q) \rightarrow \text{final state of flavour } f$$

has the general structure

$$(2.1.6) \quad M_f = j_\mu^{\text{em}} \frac{-g^{\mu\nu}}{q^2} j_\nu^{\text{em}} + j_\mu^{\text{nc}} \frac{-g^{\mu\nu} + q^\mu q^\nu / m_Z^2}{q^2 - m_Z^2 + i m_Z \Gamma_Z} j_\nu^{\text{nc}}$$

where Γ_Z is the total decay width of the Z^0 -boson and j_μ^{em} and j_μ^{nc} (j_μ^{nc}) refer, respectively, to the electromagnetic and weak leptonic (hadronic) currents. Since we are neglecting the

electron mass, (leptonic) current conservation tells us that there is no contribution coming from the term $q^\mu q^\nu / m_Z^2$ of the Z^0 -propagator. We find it convenient to split up the currents into

their vector and axial vector parts thereby extracting the relative strength of the electromagnetic and weak couplings, respectively:

$$(2.1.7) \quad j_\mu^{\text{em}} = e Q_e j_\mu^V, \quad j_\mu^{\text{nc}} = e m_Z \sqrt{g} [v j_\mu^V - a j_\mu^A]$$

$j_\mu^{\text{em}} = e Q_f j_\mu^V$, $j_\mu^{\text{nc}} = e m_Z \sqrt{g} [v_f j_\mu^V - a_f j_\mu^A]$
Here the same vector current $j_\mu^V(j_\mu^V)$ appears in j_μ^{em} and j_μ^{nc} (j_μ^{nc}). Let us define $x_Z(q^2)$, the ratio of the Z^0 -propagator to the photon propagator times a coupling strength factor:

$$(2.1.8) \quad x_Z(q^2) = g m_Z^2 \frac{q^2}{q^2 - m_Z^2 + i m_Z \Gamma_Z}$$

The matrix element for e^-e^+ annihilation into a final state of flavour f will then be

(2.1.9)

$$M_f = -\frac{e^2}{q^2} \left\{ Q_e Q_f j_\mu^V j_\mu^V + x_Z(q^2) [v j_\mu^V - a j_\mu^A] [v_f j_\mu^V - a_f j_\mu^A] \right\}$$

$$M_f = -\frac{e^2}{q^2} \sum_{i,j=V,A} x_{ij}^f j_i^\mu j_j^\mu$$

where we have introduced the elements x_{ij}^f in the last step, e.g.

$$(2.1.10) \quad x_{VV}^f = Q_e Q_f + v v_f x_Z(q^2)$$

If we square (2.1.10) and sum over (not necessary all) final spins we obtain

$$(2.1.11) \quad \frac{2}{e} \frac{q^2}{4} \sum_{\text{spins}} |M_f|^2 = \sum_{i,k,j,l=V,A} x_{ij}^f x_{kl}^{f*} L_{ik}^{\mu\nu} H_{jl}^{\mu\nu}$$

where $(i,j,k,l=V,A)$

$$(2.1.12) \quad L_{ik}^{\mu\nu} H_{jl}^{\mu\nu} = \left[\frac{2}{q^2} j_i^\mu j_k^{\nu*} \right] \left[\sum_{\text{spins}} j_j^\mu j_l^{\nu*} \right]$$

We now define four lepton tensors $L_{\mu\nu}^I$ by

$$\begin{aligned} L_1^{\mu\nu} &= \frac{1}{2} (L_{VV}^{\mu\nu} + L_{AA}^{\mu\nu}) \\ L_2^{\mu\nu} &= \frac{1}{2} (L_{VV}^{\mu\nu} - L_{AA}^{\mu\nu}) \\ L_3^{\mu\nu} &= \frac{1}{2} (L_{VA}^{\mu\nu} - L_{AV}^{\mu\nu}) \\ L_4^{\mu\nu} &= \frac{1}{2} (L_{VA}^{\mu\nu} + L_{AV}^{\mu\nu}) \end{aligned}$$

and similarly four hadron tensors $H_{\mu\nu}^I$ ($I = 1, \dots, 4$).

For later use we introduce what we call the amplitude $T_{\mu\beta}$ for the 3-parton process $\gamma, Z^0(q^2) \rightarrow q(q_1) \bar{q}(q_2) G(q_3)$. If we write the hadronic vector (axial vector) current in the form

$$(X = V, A)$$

$$\begin{aligned} J_\mu^X &= \langle G(q_3) \bar{q}(q_2) q(q_1) | J_\mu^X(0) | 0 \rangle \\ &= \bar{u}(q_1) T_{\mu\beta}^X(q_1, q_2) v(q_2) \epsilon^{\beta*}(q_3) \end{aligned}$$

we obtain

$$H_{\mu\nu}^{XY} = -\text{Tr } \gamma_{q_1} T_{\mu\beta}^X \gamma_{q_2} \gamma_\nu \frac{\gamma^\beta}{2}.$$

Regrouping the 16 terms arising on the RHS of (2.1.11) we find

$$(2.1.16) \quad \frac{2q^2}{4} \frac{Z}{\text{spins}} |M_f|^2 = \sum_{I, S=1}^4 g_{IS}^f(q^2) L_{\mu\nu}^I H_{\mu\nu}^S$$

The coefficients g_{IS}^f depend only on $s = q^2$ and on the coupling constants entering (2.1.9). They contain all the information on γ - Z^0 interference and the Z^0 -boson itself. They are certain linear combinations of the X_{ij}^f and are given explicitly in table 1.

The differential cross section for an arbitrary N -particle final state of flavour¹ f summed over (not necessarily all) final

¹The total hadronic cross section is given by: $\sigma = \sum_{f=1}^{n_f} \sigma_f$

spin orientations and for arbitrary polarized beams is now given

by ($s = q^2$):

$$\begin{aligned} (2.1.17) \quad d^3N \sigma_f &= \frac{1}{4 p_- \cdot p_+} d\text{PS}^{(N)} \sum_{\text{spins}} |M_f|^2 \\ &= \left[\frac{2\pi\alpha}{s} \right]^2 \left[\sum_{r,t=1}^4 g_{rt}^f(s) L_r^{\mu\nu} H_{\mu\nu}^t \right] d\text{PS}^{(N)} \end{aligned}$$

where $d\text{PS}^{(N)}$ is the N -particle phase space

$$(2.1.18) \quad d\text{PS}^{(N)} = (2\pi)^4 \delta^4 \left\{ \sum_{i=1}^N p_i - q \right\} \frac{N}{(2\pi)^3} \frac{d^3 p_i}{2 E_i}$$

We notice that the cross section has already a rather

factorized form: all flavour and neutral current model

dependences are comprised in the energy dependent electro-weak coupling matrix $g_{rt}^f(s)$. As we will see in the next section, the angular correlations between the initial and final state as well as the beam polarization parameters can be disentangled from the contractions $L_r^{\mu\nu} H_{\mu\nu}^t$.

Furthermore we observe that $N_C g_{11}^f$ is nothing else than the ratio

$$(2.1.19) \quad N_C g_{11}^f = R_f = \frac{\sigma(e^+e^- \rightarrow \gamma, Z \rightarrow f\bar{f})}{\sigma(e^+e^- \rightarrow \gamma \rightarrow \mu^+\mu^-)} = \frac{\sigma_2^f}{\sigma_{\text{pt}}}$$

of the total cross section for the indicated binary reaction.

Here

$$(2.1.20) \quad \sigma_{\text{pt}} = \frac{4\pi\alpha^2}{3s}$$

It is then sensible to introduce normalized coupling coefficients

$$(2.1.21) \quad A_{rt}^f(s) = \frac{g_{rt}^f(s)}{g_{11}^f(s)}$$

most of which are related to angular asymmetries (see the next section) and can be determined thereby.

In the 1γ -exchange approximation all the coefficients g_{rt}^f

vanish except for

$$(2.1.22) \quad g_{11}^f = g_{12}^f = g_{21}^f = g_{22}^f = Q_f^2$$

so that

$$(2.1.23) \quad \sum_{r,t=1}^4 g_{rt}^f(s) L_{\mu\nu}^{\mu\nu} H_{\mu\nu}^t \rightarrow Q_f^2 L_{\mu\nu}^{\mu\nu} H_{\mu\nu}^{vv}$$

Note that the above cross section formula is valid for an arbitrary N- parton final state. For later use we find it convenient to have the cross section written in the form

$$(2.1.24) \quad (2\pi)^2 \frac{d^3N \sigma^f}{\sigma_{pt}} = g_{11}^f(s) \sum_{r,t=1}^4 A_{rt}^f(s) \frac{3}{4} L_{\mu\nu}^{\mu\nu} H_{\mu\nu}^t \frac{2}{s} (2\pi)^3 d\text{ps}(N)$$

2.2 Angular Correlations and beam polarization dependences

We consider now the beam-event angular correlations that arise from the contraction of the lepton and hadron tensors. In order to disentangle the various angular dependences we find it convenient to shift all angular dependences into the explicitly known lepton tensors. The hadron tensors then become independent of the overall spatial orientation in space of a given final state configuration. Following, e.g. /14,69,77/, we introduce two different oriented center-of-mass frames $S' = Ox'y'z'$ and $S = Oxyz$. S' is convenient for the evaluation of the lepton tensors. Its axis are such that $\vec{p}_-$ is in the Oz' direction and a possible transverse electron polarization fixes the Ox' direction. We call S' the laboratory beam frame. On the other hand S is determined by the final hadron momenta and hence called event frame. In order to fix S any two noncollinear final state momenta can be chosen (see below). The relative orientation of S' relative to S is determined by the 3 Euler angles (θ, φ, χ) , see

Fig. 3.

To evaluate the quantities $L_{\mu\nu}^{\mu\nu} H_{\mu\nu}^t$ appearing in the cross section formula (2.1.24) there are two possibilities. Notice first that for massless leptons current conservation implies

$$(2.2.1) \quad q^\mu L_{\mu\nu}^{\mu\nu} = 0 \quad (q = p_- + p_+)$$

and hence in any cms frame ($\vec{q} = 0$) we have

$$(2.2.2) \quad L_{00}^{\mu\nu} = L_{0n}^{\mu\nu} = L_{m0}^{\mu\nu} = 0 \quad (m, n = 1, 2, 3).$$

Thus $L_{\mu\nu}^{\mu\nu} H_{\mu\nu}^t$ reduce to the rotationally invariant quantities $L_{mn}^{\mu\nu} H_{mn}^t$ which can be evaluated either in S' or in S . In the first case we get simple expressions for the lepton tensors¹, whereas

$$(2.2.9) \quad s = (p_- + p_+)^2 \gg 4 m_e^2$$

(2.2.6) reduces to

$$(2.2.10) \quad \rho_{\pm} = \frac{1}{2} \gamma p_{\pm} (1 - \gamma_5 \vec{\gamma} \cdot \vec{\xi}_1^{\pm} \pm h^{\pm} \gamma_5)$$

Here we have introduced the longitudinal (h) and transverse ($\vec{\xi}_1$) components of $\vec{\xi}$ with respect to $\vec{p} = \vec{p}'/|\vec{p}'|$:

$$(2.2.11) \quad h^{\pm} = \vec{\xi}^{\pm} \cdot \vec{p}_{\pm}, \quad \vec{\xi}_1^{\pm} = \vec{\xi}^{\pm} - h^{\pm} \vec{p}_{\pm}.$$

We are now in the position to evaluate the lepton tensors $L_r^{\mu\nu}$ in the laboratory beam frame $S' = O x' y' z'$ (Fig.3). In this frame

$$(2.2.12) \quad -h^{\pm} = \pm \xi_z^{\pm}, \quad \vec{\xi}_1^{\pm} = (\xi_x^{\pm}, \xi_y^{\pm}, 0).$$

Since we neglect terms of order $4 m_e^2/s$, we find that the only nonvanishing components of

$$(2.2.13) \quad L'_{\mu\nu} \equiv L_{\mu\nu}(S')$$

are those for $\mu, \nu = 1, 2$ given explicitly by /96/

$$\begin{aligned} (2.2.14) \quad L_1'^{11} &= L_1'^{22} &= (1 - h^- h^+) \\ L_1'^{12} &= -L_1'^{21} &= -i (h^- - h^+) \\ L_2'^{11} &= -L_2'^{22} &= -(\xi_x^-, \xi_x^+ - \xi_y^-, \xi_y^+) \\ L_2'^{12} &= L_2'^{21} &= -(\xi_x^-, \xi_y^+ + \xi_y^-, \xi_x^+) \\ L_3'^{11} &= -L_3'^{22} &= (\xi_x^-, \xi_y^+ + \xi_y^-, \xi_x^+) \\ L_3'^{12} &= L_3'^{21} &= -(\xi_x^-, \xi_x^+ - \xi_y^-, \xi_y^+) \\ L_4'^{11} &= L_4'^{22} &= (h^- - h^+) \\ L_4'^{12} &= -L_4'^{21} &= -i (1 - h^- h^+). \end{aligned}$$

We find that the longitudinal (transverse) beam polarization

contributes only to $L_1', L_4' (L_2', L_3')$; this makes the definitions (2.1.13) advantageous. Of course, the appearance of the helicity factors $(1 - h^- h^+)$ and $(h^+ - h^-)$ in L_1' and L_4' is a consequence of the helicity conservation at the $(e^+ e^-)$ and $(e^+ e^-)$ vertices.

the hadron tensors depend explicitly on $(\theta, \varphi, \kappa)$. In the second case the dependence on $(\theta, \varphi, \kappa)$ comes in through the lepton tensors, while the hadron tensors depend only on internal variables.

We will proceed as follows: First we calculate the lepton tensors in S' . Then we rotate them to the event frame S . Introducing appropriate combinations of the hadron tensors allows us to disentangle all relative angular dependences. The last step is to rewrite the phase-space differential in terms of the Euler angles and scalar products of hadron momenta.

Lepton Tensors

To evaluate the various lepton tensors defined in (2.1.13) we first consider the lepton tensors defined in (2.1.12):

$$(2.2.3) \quad L_{ij}^{\mu\nu} = \frac{2}{q} j^{\mu} j^{\nu*} \quad i, j = V, A$$

Since the leptonic currents are simply given by

$$(2.2.4) \quad j^{\mu} = \bar{v}(p_+) \gamma^{\mu} u(p_-); \quad j^{\mu} = \bar{v}(p_+) \gamma^{\mu} \gamma_5 u(p_-)$$

we get

$$(2.2.5) \quad L_{ij}^{\mu\nu} = \frac{2}{q} \text{Tr} \gamma^{\mu} \gamma^{\nu} \rho_+ \gamma^{\mu} \gamma^{\nu} \rho_+$$

where

$$(2.2.6) \quad \rho_{\pm} = -(-\gamma p_{\pm} + m_e) \frac{1}{2} (1 + \gamma_5 \gamma s_{\pm})$$

and

$$(2.2.7) \quad r_i = \begin{cases} 1 & \text{for } i=V \\ \gamma_5 & \text{for } i=A \end{cases}$$

Here $p_{\pm}$ and $s_{\pm}$ are the momentum and polarization 4- vectors of $e^{\pm}$ satisfying

$$(2.2.8) \quad p_{\pm}^2 = m_e^2; \quad p_{\pm} s_{\pm} = 0; \quad s_{\pm}^2 = -|\vec{s}_{\pm}|^2 < -1;$$

$\vec{s}_{\pm}$ denote the rest-frame polarization vectors. In the high-energy limit

Only $e_R^+ e_L^-$ and $e_L^+ e_R^-$ couple, since we neglect mass terms.

The next step is to rotate the lepton tensors $L_{mn}^{\prime I}$ to the event frame S. Since S is obtained from S' by the rotation $R = R(\varphi, \theta, \chi)$ through Euler angles φ, θ, χ (see Fig. 3), i.e.

$$(2.2.15) \quad R = \begin{bmatrix} \cos\varphi \cos\theta \cos\chi - \sin\varphi \sin\chi & -\cos\varphi \cos\theta \sin\chi - \sin\varphi \cos\chi & \sin\theta \cos\varphi \\ \sin\varphi \cos\theta \cos\chi + \cos\varphi \sin\chi & -\sin\varphi \cos\theta \sin\chi + \cos\varphi \cos\chi & \sin\theta \sin\varphi \\ -\sin\theta \cos\chi & \sin\theta \sin\chi & \cos\theta \end{bmatrix}$$

we get from the transformation law of a tensor:

$$(2.2.16) \quad L_{\nu}^{mn}(\varphi, \theta, \chi) = R^{mk}(\varphi, \theta, \chi) L_{\nu}^{kl} R^{nl}(\varphi, \theta, \chi), \quad (m, n=1, 2, 3).$$

Since the expressions for L_{ν}^{mn} are rather lengthy we give the more

compact expressions for the contraction of L_{ν}^{mn} with the hadron tensors

H_{mn}^t :

$$(2.2.17) \quad L_{\nu}^{mn} H_{mn}^t = \frac{1}{2} (L_{\nu}^{11} + L_{\nu}^{22})(H_{11}^t + H_{22}^t) + \frac{1}{2} (L_{\nu}^{11} - L_{\nu}^{22})(H_{11}^t - H_{22}^t) + L_{\nu}^{33} H_{33}^t + \left\{ \frac{1}{2} (L_{\nu}^{12} + L_{\nu}^{21})(H_{12}^t + H_{21}^t) + \frac{1}{2} (L_{\nu}^{12} - L_{\nu}^{21})(H_{12}^t - H_{21}^t) \right\} + \left\{ 2, 3 \right\} + \left\{ 3, 1 \right\}.$$

This motivates us to define certain combinations of the space-space components of H_{mn}^t .

Hadron Tensors

They are defined in (2.1.12) and in (2.1.13). As we have just seen, in any cms system only the space components of the four hadron tensors H_{mn}^t contribute to the products $L_{\nu}^{\mu\nu} H_{\mu\nu}^t$ and hence to the cross section. Also, since the hadron tensors are hermitean by definition, there are in general nine real independent quantities which describe each H_{mn}^t . Corresponding to the decomposition (2.2.17)

we introduce² the following nine "cross sections" $\sigma_{\alpha}(H^I)$ of the nine space-space components of $H_{mn}^I = H_{mn}^{I\mu\nu}$ ($m, n=1, 2, 3$; $\alpha=1, \dots, 9$):

$$(2.2.18) \quad \begin{aligned} J^{-1} \sigma_1(H^I) &= H_{++} + H_{--} &= H_{11} + H_{22} \\ J^{-1} \sigma_2(H^I) &= H_{00} &= H_{33} \\ J^{-1} \sigma_3(H^I) &= \frac{1}{2} (H_{+-} + H_{-+}) &= \frac{1}{2} (-H_{11} + H_{22}) \\ J^{-1} \sigma_4(H^I) &= \frac{1}{2} \operatorname{Re} (H_{+0} - H_{-0}) &= -\frac{1}{2\sqrt{2}} (H_{31} + H_{13}) \\ J^{-1} \sigma_5(H^I) &= \frac{1}{2} \operatorname{Im} (H_{+0} - H_{-0}) &= \frac{1}{2\sqrt{2}} (H_{31} - H_{13}) \\ J^{-1} \sigma_6(H^I) &= H_{+-} - H_{-+} &= -i (H_{12} - H_{21}) \\ J^{-1} \sigma_7(H^I) &= \frac{1}{2} \operatorname{Re} (H_{+0} + H_{-0}) &= \frac{1}{2\sqrt{2}} (H_{23} - H_{32}) \\ J^{-1} \sigma_8(H^I) &= \frac{1}{2} \operatorname{Im} (H_{+0} + H_{-0}) &= -\frac{1}{2\sqrt{2}} (H_{23} + H_{32}) \\ J^{-1} \sigma_9(H^I) &= \operatorname{Im} H_{+-} &= -\frac{1}{2} (H_{12} + H_{21}) \end{aligned}$$

Here

$$(2.2.19) \quad H_{\rho\sigma}^I = \epsilon_{\rho}^m H_{mn}^I \epsilon_{\sigma}^{n*}, \quad (\rho, \sigma = \pm, 0)$$

with

$$(2.2.20) \quad \epsilon_{+}^m = (-1, -i, 0)/\sqrt{2}, \quad \epsilon_{-}^m = (1, -i, 0)/\sqrt{2}, \quad \epsilon_0^m = (0, 0, 1).$$

In terms of the polarization vectors the helicity cross sections σ_{α} have the following interpretation: σ_1 (σ_2) is the cross section for unpolarized transverse (longitudinal polarized) virtual bosons γ or Z with helicity axis Oz. σ_3 (σ_4) correspond to the interference of helicity +1 and helicity -1 amplitudes (the real part of +1 and 0 interference), etc. It has become customary to denote σ_{α} for

²The normalization factor J is defined in (2.2.26); note that in 1 γ -exchange calculations the σ_{α} are sometimes (e.g. /73/) normalized by a factor $(J \sigma_{pt} N_C x Q_f^2)^{-1}$ instead of J^{-1} .

$\alpha = 1, 2, 3, 4$ as $\sigma_Y, \sigma_L, \sigma_T, \sigma_I$, respectively. In the following we will use a double notation, i.e. we will use both $\alpha=1, 2, 3, 4$ and $\alpha=U, L, T, I$.

The definition in terms of polarization vectors will prove to be useful when generalizing to final state polarization, see section 2.5.

Six of the above cross sections $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_8, \sigma_9)$ are symmetric under $m \leftrightarrow n$ whereas three $(\sigma_5, \sigma_6, \sigma_7)$ are antisymmetric. Since for any hermitean tensor H_{mn} its (anti-) symmetric part is equal to its (imaginary) real part

$$(2.2.21) \quad H_{mn} = H_{mn}^{\text{sym}} + H_{mn}^{\text{anti}} = \text{Re } H_{mn} + i \text{Im } H_{mn}$$

a (imaginary) real hadron tensor may contribute only to $(\sigma_5, \sigma_6, \sigma_7)$ or $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_8, \sigma_9)$, respectively.

Angular Correlations

We are now in position to give explicit expressions for the contractions (2.2.17). Introducing the angular coefficients $d_{K\alpha}(\theta, N)$ of table 2 and defining the quantities $\sigma_K(H^t)$ by

$$(2.2.22) \quad \sigma_K(H^t) = \sum_{\alpha=1}^9 d_{K\alpha}(\theta, N) \sigma_{\alpha}(H^t), \quad (K = A, B, C, D)$$

we obtain the following results:

$$(2.2.23) \quad \begin{aligned} J \frac{3}{4} L_{11}^{mn} H_{mn}^t &= Z_1 \sigma_A(H^t) + Z_2 \sigma_D(H^t) \\ J \frac{3}{4} L_{22}^{mn} H_{mn}^t &= X(\varphi) \sigma_B(H^t) + Y(\varphi) \sigma_C(H^t) \\ J \frac{3}{4} L_{33}^{mn} H_{mn}^t &= X(\varphi) \sigma_C(H^t) - Y(\varphi) \sigma_B(H^t) \\ J \frac{3}{4} L_{44}^{mn} H_{mn}^t &= Z_1 \sigma_D(H^t) + Z_2 \sigma_A(H^t) \end{aligned}$$

where

$$(2.2.24) \quad \begin{aligned} X(\varphi) &= X_1 \cos 2\varphi + X_2 \sin 2\varphi \\ Y(\varphi) &= -X_1 \sin 2\varphi + X_2 \cos 2\varphi \end{aligned}$$

and

$$(2.2.25) \quad \begin{aligned} Z_1 &= 1 - h^+ h^-, & X_1 &= \xi_X^+ \xi_X^- + \xi_Y^+ \xi_Y^-, \\ Z_2 &= h^+ h^-, & X_2 &= \xi_X^+ \xi_Y^- + \xi_X^- \xi_Y^+ \end{aligned}$$

Let us examine some special cases of beam polarization:

- (i) for longitudinal polarization of helicity $\lambda^{\pm} (= \pm \frac{1}{2})$ we have $h^{\pm} = 2\lambda^{\pm}$ and $X(\varphi) = Y(\varphi) = 0$;
- (ii) for natural transverse polarization we set $h^{\pm} = \xi_X^{\pm} = 0$ and $\xi_Y^{\pm} = \pm p$, therefore $Z_1 = 1, Z_2 = 0, X(\varphi) = p^2 \cos 2\varphi, Y(\varphi) = p^2 \sin 2\varphi$;
- (iii) for unpolarized beams: $Z_1 = 1, Z_2 = X(\varphi) = Y(\varphi) = 0$.

This completes our task of decoupling the angular dependences from of the products $L_{\mu\nu}^{\mu\nu} H^{\mu\nu t}$: in (2.2.23) the quantities $X(\varphi), Y(\varphi)$ and $d_{K\alpha}(\theta, N)$ carry the whole angular dependence whereas the cross sections $\sigma_{\alpha}(H^t)$ are defined in the event frame S and thus do not contain any dependence on φ, θ and N .

Phase Space

Before we can write down the cross section formula we have to rewrite the phase- space differential (2.1.18) in terms of the Euler angles and a set of 3N-7 final state variables $(x) \equiv (x_1, \dots, x_{3N-7})$.

These could be chosen as e.g. scaled energies and/or relative angles $q_i, q_k \equiv \cos \theta_{ik}$. For this change of variables we write:

$$(2.2.26) \quad \frac{2}{s} (2\pi)^3 dPS(N) = J d\varphi dN d\cos\theta d(x)$$

where J is the corresponding Jacobian which does not depend on φ, θ and N .

By choosing the event frame as indicated in Fig. 4 we easily find that the 3-parton Jacobian is given by

$$(2.2.27) \quad J = \frac{1}{64\pi^2}$$

and that the x_i can be defined by the parton momenta q_i

$$(2.2.28) \quad x_i = \frac{2q_i \cdot q}{s} \quad (i = 1, 2, 3)$$

Cross Section

Using (2.2.26) and (2.2.23) in (2.1.24) we obtain /96/

$$(2.2.29) \quad \frac{(2\pi)^2}{\sigma_{pt}} \frac{d^{3N-4} \sigma^f}{d^3 N d\cos\theta d(x)} = g_{11}^f \left\{ Z_1^1 Z^1 + X(p) Z^X + Y(p) Z^Y + Z_2^2 Z^2 \right\}$$

where

$$(2.2.30) \quad \begin{aligned} Z^1 &= \sum_{t=1}^4 \left[A_{1t}^f \sigma_A(H^t) + A_{4t}^f \sigma_D(H^t) \right] \\ Z^X &= \sum_{t=1}^4 \left[A_{2t}^f \sigma_B(H^t) + A_{3t}^f \sigma_C(H^t) \right] \\ Z^Y &= \sum_{t=1}^4 \left[A_{2t}^f \sigma_C(H^t) - A_{3t}^f \sigma_B(H^t) \right] \\ Z^2 &= \sum_{t=1}^4 \left[A_{1t}^f \sigma_D(H^t) + A_{4t}^f \sigma_A(H^t) \right] \end{aligned}$$

In this form the influences of the various physical quantities on the cross section can be read off:

- (i) the angular correlations between initial and final state, which merely reflect the spin- one nature of the virtual γ and Z^0 , are contained in $X(p)$, $Y(p)$ and $d_{K\alpha}(\theta, N)$;
- (ii) the beam polarization shows up in $X(p=0)$, $Y(p=0)$ and $h^\pm$;
- (iii) the neutral current model (and flavour) dependences are summarized in the normalized coupling coefficient $A_{rt}(s)$;
- (iv) all the hadronic QCD effects are contained in $\sigma_\alpha(H^t)$.

2.3 Differential cross sections versus structure functions

We consider now the 3-parton process

$$\gamma, Z^0(q) \rightarrow q(q_1) + \bar{q}(q_2) + G(q_3)$$

in more detail. Before we can give the 3-parton cross section

formula (2.2.29) in a form more suited to experimental analysis we have to study the hadron tensors describing e^+e^- annihilation into three partons in more detail. Two ways are open:

- (a) we can analyze the covariant structure of the hadron tensors, or
- (b) we can specify the event frame and study the properties of the hadron tensors in this specific frame.

Of course, both formulations should lead to the same results,

especially tell us the number of independent quantities describing the 3- parton cross section. In case (a) we would start from the (covariant) definitions (2.1.12-15) while in (b) we would consider the helicity cross sections $\sigma_\alpha(H^t)$ (2.2.18), i.e. the (frame dependent) space- space components of the hadron tensors. The covariant form $H_{\mu\nu}$ is better suited for a delineation of the general structure of our hadron tensors and for theoretical calculations, whereas the helicity cross sections enter the experimental analysis.

Since we are considering e^+e^- annihilation at very high energies ($\sqrt{s} \approx 30$ GeV) we can neglect the quark masses ($s \gg 4m_q^2$; also $m_G=0$). The masslessness implies that all Feynman amplitudes describing the vector current $\gamma \rightarrow q\bar{q}G$ consist of an odd product of γ - matrices. Hence the axial vector current obeys the following

we conclude that $H_{\mu\nu}^{p.c.}$ ($H_{\mu\nu}^{p.v.}$) is even (odd) under C. Let us now examine the covariant structure of the p.c. and p.v. hadron tensors. Note first that current conservation at the electromagnetic vertex implies

$$(2.3.7) \quad q^\mu H_{\mu\nu}^X = H_{\mu\nu}^X q^\nu = 0 \quad (x = p.c., p.v.).$$

Further we observe that the final state contains only three independent momenta, e.g.

$$(2.3.8) \quad q_1, q_2, q = q_1 + q_2 + q_3$$

where q_1, q_2 and q_3 denote the momenta of quark, antiquark and gluon, respectively. We can now write down the most general structure of the two hadron tensors /65,90/

$$(2.3.9) \quad H_{\mu\nu}^{p.c.} = \sum_{i=1}^5 H_i H_{\mu\nu}^i$$

$$= H_1 \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + H_2 \frac{1}{q^2} q_{1\mu} q_{1\nu} + H_3 \frac{1}{q^2} q_{2\mu} q_{2\nu} + H_4 \frac{1}{q^2} (q_{1\mu} q_{2\nu} + q_{2\mu} q_{1\nu}) + H_5 \frac{1}{q^2} (q_{1\mu} q_{2\nu} - q_{2\mu} q_{1\nu})$$

and²

$$(2.3.10) \quad H_{\mu\nu}^{p.v.} = \sum_{i=6}^9 H_i H_{\mu\nu}^i$$

$$= H_6 \frac{1}{q^2} \epsilon(\mu, \nu, q, q_1) + H_7 \frac{1}{q^2} \epsilon(\mu, \nu, q, q_2) + H_8 \frac{1}{q^2} (q_{1\mu} F_\nu + F_\mu q_{1\nu}) + H_9 \frac{1}{q^2} (q_{2\mu} F_\nu + F_\mu q_{2\nu})$$

Here $F_\mu = i \epsilon(\mu, q_1, q_2, q_3)$. The symbol "..." denotes the gauge invariant completion

$$(2.3.11) \quad p = p - \frac{p \cdot q}{q^2} q$$

²(2.3.10) is valid in 4 dimensions only; see comment at the end of this chapter.

relation¹

$$(2.3.1) \quad J_{\mu\beta}^A = -\gamma_5 J_{\mu\beta}^V$$

From the definition (2.1.12) of the hadron tensors $H_{\mu\nu}^{ij}$

(i,j = V,A) we conclude immediately

$$(2.3.2) \quad H_{\mu\nu}^{VV} = H_{\mu\nu}^{AA}, \quad H_{\mu\nu}^{AV} = H_{\mu\nu}^{VA}$$

Using the definitions (2.1.13) of the hadron tensors $H_{\mu\nu}^t$ we find that two of them vanish identically:

$$(2.3.3) \quad H_{\mu\nu}^2 = H_{\mu\nu}^3 = 0$$

The remaining two hadron tensors can be distinguished by

their behaviour under parity (P) and charge conjugation (C)

transformation. The transformation behaviour under $P: \vec{x} \rightarrow -\vec{x}$ can be read off from their definitions (2.1.12,13): H_1 (H_4) is even (odd) under P, i.e.

$$(2.3.4) \quad \begin{aligned} H_{\mu\nu}^1 &\rightarrow + H_{\mu\nu}^1 \\ H_{\mu\nu}^4 &\rightarrow - H_{\mu\nu}^4 \end{aligned}$$

Therefore the cross sections built up from H_1 (H_4) are parity-conserving (parity-violating). This motivates the following definitions:

$$(2.3.5) \quad H_{\mu\nu}^{p.c.} \equiv H_{\mu\nu}^1, \quad H_{\mu\nu}^{p.v.} \equiv H_{\mu\nu}^4$$

p.c. = parity-conserving, p.v. = parity-violating

The two hadron tensors can also be classified by their behaviour under C. Using the fact

$$(2.3.6) \quad \begin{aligned} J_{\mu}^V(q_1, q_2) &\rightarrow -J_{\mu}^V(q_2, q_1) \\ J_{\mu}^A(q_1, q_2) &\rightarrow +J_{\mu}^A(q_2, q_1) \end{aligned}$$

¹We mention already here that this identity is no longer true in general in n dimensions. For the consequences see §5.

and $\epsilon(\mu, \nu, q, q_1)$ stands for $\epsilon_{\mu\nu\alpha\beta} q^\alpha q_1^\beta$ etc. .

At first sight it is surprising that there are no two further antisymmetric $(\mu \rightarrow \nu)$ covariants in (2.3.10) with the structure $(\bar{q}_1 \mu \bar{F}_\nu - \bar{F}_\mu \bar{q}_{1\nu})$. However, these two are reduced to the antisymmetric covariants already occurring in (2.3.10) by use of Schouten's identity³

$$(2.3.12) \quad k_{\mu_1} \epsilon(\mu_2 \mu_3 \mu_4 \mu_5) + \text{cycl.}(\mu_i) = 0$$

From the hermiticity of the hadron tensors $H_{\mu\nu}^x = H_{\nu\mu}^{x*}$ we conclude that $H_1, \dots, H_4, H_6$ and H_7 are real and H_5, H_8 and H_9 are imaginary.

We now turn to the (physical) helicity cross sections. Since there are only nine independent invariant structure functions, the (general) cross section must be describable by nine independent

(helicity) cross sections σ_α . We first observe that together with $H_{\mu\nu}^2$ and $H_{\mu\nu}^3$ also $\sigma_\alpha(H^2)$ and $\sigma_\alpha(H^3)$ are zero. Corresponding to the definitions for the hadron tensors we set:

$$(2.3.13) \quad \begin{aligned} \sigma_\alpha^{\text{p.c.}} &\equiv \sigma_\alpha(H^1), & \alpha &= 1, \dots, 9 \\ \sigma_\alpha^{\text{p.v.}} &\equiv \sigma_\alpha(H^4), & \alpha &= 1, \dots, 9. \end{aligned}$$

Before we specify the event frame we observe that 3- particle final states are planar events since their three momenta always lie in a plane, the so called event- plane. Depending on the way we relate this event frame to the event plane two types of event coordinate systems have proven to be useful in the analysis of

³ We note here that this is a pure 4- dimensional identity. When calculating the p.v order α^2 corrections within dimensional regularization, (2.3.12) is no longer valid.

3-jet events in e^+e^- annihilation /63.96/. In the helicity system the event z- axis is chosen to lie in the event plane whereas in the transversity frame the z- axis is chosen normal to the event plane. Correspondingly the Euler angles $(\varphi, \theta, \kappa)$ connecting the beam system S' with the event frame S (Fig.3) have different physical meaning in the two cases (although their mathematical definition is always given according to Fig. 3) :

(a) Helicity system. φ is the azimuthal angle around the z'- beam axis between the storage ring plane $(x'-z')$ and the "scattering plane" $(z'-z)$; κ is the azimuthal angle around the z- axis between the "scattering plane" $(z'-z)$ and the event plane $(z-x)$; θ is the polar angle between the z'- beam axis and the z- event axis.

(b) Transversity system. The Euler angles are denoted by $(\bar{\varphi}, \bar{\kappa}, \bar{\theta})$: $\bar{\varphi}$ is the azimuthal angle around the z'- beam axis between the storage ring plane $(x'-z')$ and the $(z'-z)$ plane; $\bar{\kappa}$ is the azimuthal angle around the z- axis between the $(z'-z)$ plane and the plane $(x-z)$ perpendicular to the event plane; $\bar{\theta}$ is the polar-angle between the z'- beam axis and the normal Oz to the event plane.

In Fig. 4 we show possible choices of a helicity- and a transversity- frame. In order to determine the Euler angles unambiguously we have to fix the event frame (helicity or transversity) in the final state. One choice is to say the event z- axis is along the momentum of the most energetic jet while the second most energetic jet momentum determines the event y- axis. More about this choice will be discussed in §7.

We notice that the two systems are in turn connected by an

Euler rotation $R(-\frac{\pi}{2}, -\frac{\pi}{2}, 0)$ which allows one to relate the hadron tensor components in the two event frames. One has /96/:

$$(2.3.14) \quad \begin{aligned} \bar{\sigma}_U &\equiv \bar{\sigma}_1 = \frac{1}{2} \sigma_1 + \sigma_2 - \sigma_3 \\ \bar{\sigma}_L &\equiv \bar{\sigma}_2 = \frac{1}{2} \sigma_1 + \sigma_3 \\ \bar{\sigma}_T &\equiv \bar{\sigma}_3 = \frac{1}{4} \sigma_1 - \frac{1}{2} (\sigma_2 + \sigma_3) \\ \bar{\sigma}_I &\equiv \bar{\sigma}_4 = \sigma_8 \\ \bar{\sigma}_5 &= \sigma_7 \\ \bar{\sigma}_6 &= 2\sqrt{2} \sigma_5 \\ \bar{\sigma}_7 &= \frac{1}{2\sqrt{2}} \sigma_6 \\ \bar{\sigma}_8 &= \frac{1}{\sqrt{2}} \sigma_9 \\ \bar{\sigma}_9 &= \sqrt{2} \sigma_4 \end{aligned}$$

where the $\bar{\sigma}_\alpha(H^t)$ refer to the hadron tensor components in the transversity frame.

It is instructive to see how the number nine of independent cross sections comes out in the frame-dependent analysis. Since 3-parton events lie in a plane the final state must have a symmetry with respect to this plane. It is of course sufficient to analyse this additional mirror symmetry and the constraints it is inducing in one frame. For this we choose the helicity frame. Here the event plane is chosen as the Oxz- plane of the event frame $S = Oxyz$. So reflection in this plane means $\vec{y} \rightarrow -\vec{y}$. Under this reflection a vector $\vec{J}$ and an axial- vector $\vec{K}$ transform according to

$$(2.3.15) \quad \begin{aligned} (J_x, J_y, J_z) &\rightarrow (J_x, -J_y, J_z) \\ (A_x, A_y, A_z) &\rightarrow (-A_x, A_y, -A_z). \end{aligned}$$

Invariance under this symmetry operation then implies for the hadron tensors H_r^{mn} in (2.1.12,13) (remember that $H_2 = H_3 = 0$):

$$(2.3.16) \quad \begin{aligned} H_1^{12} &= H_1^{21} = H_1^{23} = H_1^{32} = 0 \\ H_4^{11} &= H_4^{22} = H_4^{33} = H_4^{13} = H_4^{31} = 0. \end{aligned}$$

According to (2.2.18) this means that

$$(2.3.17) \quad \sigma_6(H_1) = \sigma_7(H_1) = \sigma_8(H_1) = \sigma_9(H_1) = 0$$

$$\sigma_1(H_4) = \sigma_2(H_4) = \sigma_3(H_4) = \sigma_4(H_4) = \sigma_5(H_4) = 0$$

We find that H_1 and H_4 contribute complementary⁴ to different cross section and that altogether there are indeed nine independent cross sections.

Let us summarize the simplifications for the four hadron tensors H^t obtained so far: first, two of them ($H_{\mu\nu}^2$ and $H_{\mu\nu}^3$) vanish identically because we neglect quark masses; second, the remaining two hadron tensors $H_{\mu\nu}^1$ and $H_{\mu\nu}^4$ are described by 5 and 4 quantities (structure functions or helicity cross sections), respectively; this is due to the fact that we are considering 3-parton events only.

What remains to be done is to give the connections between the helicity cross sections σ_α and the structure functions H_i . We will give the relations between σ_α and H_i in the (event cms $\vec{q} = 0$) frame where $\vec{q}_1$ is along the (event) z- axis and $\vec{q}_2$ lies in the positive (x-z)- half- plane⁵. Here we have ($q_i^2 = 0$):

⁴ Since $\sigma_\alpha \equiv \sigma_\alpha^{\text{p.c.}}$ for $\alpha=1, \dots, 5$ and $\sigma_\alpha \equiv \sigma_\alpha^{\text{p.v.}}$ for $\alpha=6, \dots, 9$, we can drop the indices p.c. and p.v. in the following.

⁵ Relations in the other 5 possible (helicity) event frames can be obtained in complete analogy.

(2.3.18)

$$\begin{aligned}\vec{q}_1 &= E_1 (0, 0, 1) \\ \vec{q}_2 &= E_2 (+\sin\theta_{12}, 0, \cos\theta_{12}) \\ \vec{q}_3 &= E_3 (-\sin\theta_{13}, 0, \cos\theta_{13})\end{aligned}$$

where θ_{ij} is the angle between particle i and j :

$$(2.3.19) \quad \cos\theta_{ij} = \frac{(1 - x_i)(1 - x_j) - (1 - x_k)}{x_i x_j} \quad i \neq j \neq k$$

$$\sin\theta_{ij} = \frac{2\sqrt{X}}{x_i x_j}, \quad X = \prod_{i=1}^3 (1 - x_i)$$

and x_i is the scaled energy of parton i :

$$(2.3.20) \quad x_i = \frac{2E_i}{\sqrt{q^2}}, \quad x_1 + x_2 + x_3 = 2$$

which is related to the (scaled) invariant scalar products of the parton momenta by

$$(2.3.21) \quad x_i = 1 - Y_{jk}, \quad Y_{jk} = \frac{s_{jk}}{q^2} = \frac{2q_j \cdot q_k}{q^2}$$

Using the definitions (2.2.18) of σ_α and the explicit

representation (2.3.18) in the general expansion (2.3.9,10), it is straightforward to show that ($J = 1/(64\pi^2)$, see (2.2.27)):

$$(2.3.22) \quad \begin{aligned}J^{-1} \sigma_1^{p.c.} &= -2H_1 + \frac{X}{x_1^2} H_3 \\ J^{-1} \sigma_2^{p.c.} &= -H_1 + \frac{x_1}{4} H_2 + \frac{x_2}{2} \cos\theta_{12} \left(\frac{x_2}{2} \cos\theta_{12} H_3 + x_1 H_4 \right) \\ J^{-1} \sigma_3^{p.c.} &= -\frac{X}{2x_1^2} H_3\end{aligned}$$

$$\begin{aligned}J^{-1} \sigma_4 &= -\frac{1}{4} \sqrt{X} (H_4 + \frac{x_2}{x_1} \cos\theta_{12} H_3) \\ J^{-1} \sigma_5^{p.c.} &= -\frac{i}{4} \sqrt{X} H_5\end{aligned}$$

Similarly we get

$$(2.3.23) \quad \begin{aligned}J^{-1} \sigma_6^{p.v.} &= x_1 H_6 + x_2 \cos\theta_{12} H_7 \\ J^{-1} \sigma_7^{p.v.} &= \frac{x_2}{2\sqrt{2}} \sin\theta_{12} H_7 \\ J^{-1} \sigma_8^{p.v.} &= \frac{i}{8} \sqrt{X} (x_1 H_8 + \frac{x_2}{4} \cos\theta_{12} H_9) \\ J^{-1} \sigma_9^{p.v.} &= \frac{i}{4} x_2 \sin\theta_{12} \sqrt{X} H_9\end{aligned}$$

2.4 The 3-parton cross section

Collecting the results obtained so far the 3-parton cross section (2.2.29) simplifies to

$$\begin{aligned}(2.4.1) \quad & \frac{(2\pi)^2}{911\sigma_{pt}} \frac{d^5\sigma}{d^3p \, d\Omega \, d\cos\theta \, dx_1 \, dx_2} = \\ & Z_1 \left[A_{11} \sigma_A^{p.c.} + A_{41} \sigma_D^{p.c.} + A_{14} \sigma_A^{p.v.} + A_{44} \sigma_D^{p.v.} \right] \\ & + X(\varphi) \left[A_{21} \sigma_B^{p.c.} + A_{31} \sigma_C^{p.c.} + A_{24} \sigma_B^{p.v.} + A_{34} \sigma_C^{p.v.} \right] \\ & + Y(\varphi) \left[A_{21} \sigma_C^{p.c.} - A_{31} \sigma_B^{p.c.} + A_{24} \sigma_C^{p.v.} - A_{34} \sigma_B^{p.v.} \right] \\ & + Z_2 \left[A_{11} \sigma_D^{p.c.} + A_{41} \sigma_A^{p.c.} + A_{14} \sigma_D^{p.v.} + A_{44} \sigma_A^{p.v.} \right]\end{aligned}$$

where $Z_1, Z_2, X(\varphi)$ and $Y(\varphi)$ are given in (2.2.24,25). The σ_K ($K = A, \dots, D$) are given explicitly by

$$\begin{aligned}(2.4.2) \quad \sigma_A^{p.c.} &= \frac{3}{8} (1 + \cos^2\theta) \sigma_U^{p.c.} + \frac{3}{4} \sin^2\theta \sigma_L^{p.c.} \\ &+ \frac{3}{4} \sin^2\theta \cos 2\kappa \sigma_T^{p.c.} - \frac{3}{2\sqrt{2}} \sin 2\theta \cos \kappa \sigma_I^{p.c.} \\ \sigma_B^{p.c.} &= \frac{3}{8} \sin^2\theta (\sigma_U^{p.c.} - 2 \sigma_L^{p.c.}) \\ &+ \frac{3}{4} (1 + \cos^2\theta) \cos 2\kappa \sigma_T^{p.c.} + \frac{3}{2\sqrt{2}} \sin 2\theta \cos \kappa \sigma_I^{p.c.} \\ \sigma_C^{p.c.} &= \frac{3}{2} \cos \theta \sin 2\kappa \sigma_T^{p.c.} + \frac{3}{\sqrt{2}} \sin \theta \sin \kappa \sigma_I^{p.c.} \\ \sigma_D^{p.c.} &= \frac{3}{\sqrt{2}} \sin \theta \sin \kappa \sigma_5^{p.c.}\end{aligned}$$

and

(2.4.3)

$$\begin{aligned}\sigma_A^{p.v.} &= -\frac{3}{4} \sin^2\theta \sin 2\kappa \sigma_9^{p.v.} + \frac{3}{2\sqrt{2}} \sin 2\theta \sin \kappa \sigma_8^{p.v.} \\ \sigma_B^{p.v.} &= -\frac{3}{4} (1 + \cos^2\theta) \sin 2\kappa \sigma_9^{p.v.} - \frac{3}{2\sqrt{2}} \sin 2\theta \sin \kappa \sigma_8^{p.v.} \\ \sigma_C^{p.v.} &= \frac{3}{2} \cos \theta \cos 2\kappa \sigma_9^{p.v.} + \frac{3}{\sqrt{2}} \sin \theta \cos \kappa \sigma_8^{p.v.} \\ \sigma_D^{p.v.} &= \frac{3}{4} \cos \theta \sigma_6^{p.v.} - \frac{3}{\sqrt{2}} \sin \theta \cos \kappa \sigma_7^{p.v.}\end{aligned}$$

where the helicity cross sections σ_α ($\alpha=1, \dots, 9$) are defined in

(2.2.18).

It is sometimes more convenient to have the cross section written in the form

$$(2.4.4) \quad \frac{(2\pi)^2}{g_{11}^2 \sigma_{pt}} \frac{d^5 \sigma}{d\mathbf{p} \, dN \, d\cos\theta \, dx_1 \, dx_2} =$$

$$G_{A1} \sigma_A^{p.c.} + \cos 2\varphi \left[G_{B1} \sigma_B^{p.c.} + G_{C1} \sigma_C^{p.c.} \right]$$

$$+ \sin 2\varphi \left[G_{C1} \sigma_B^{p.c.} - G_{B1} \sigma_C^{p.c.} \right]$$

$$+ G_{D4} \sigma_D^{p.v.} + G_{D1} \sigma_D^{p.c.}$$

$$+ G_{A4} \sigma_A^{p.v.} + \cos 2\varphi \left[G_{B4} \sigma_B^{p.v.} + G_{C4} \sigma_C^{p.v.} \right]$$

$$\sin 2\varphi \left[G_{C4} \sigma_B^{p.v.} - G_{B4} \sigma_C^{p.v.} \right]$$

Here we have introduced the polarization dependent couplings (we drop the flavour index f , and A_{rt} is defined in (2.1.21))

$$(2.4.5) \quad \begin{aligned} G_{Ar} &= Z_1 A_{1r} + Z_2 A_{4r} \\ G_{Br} &= X_1 A_{2r} - X_2 A_{3r} \\ G_{Cr} &= X_1 A_{3r} + X_2 A_{2r} \\ G_{Dr} &= Z_1 A_{4r} + Z_2 A_{1r} \end{aligned}$$

This representation will prove to be useful when discussing the results of the order α_s^2 calculation in section 7. Anticipating already here the fact that 3 cross sections σ_α vanish up to order α_s^2 :

$$(2.4.6) \quad \sigma_5^{p.c.} = \sigma_8^{p.v.} = \sigma_9^{p.v.} = 0,$$

so that

$$(2.4.7) \quad \sigma_D^{p.c.} = \sigma_A^{p.v.} = \sigma_B^{p.v.} = \sigma_C^{p.v.} = 0.$$

we find that up to and including the order α_s^2 the last three lines in (2.4.4) are zero.

The cross section (2.4.4) may not directly be measured at present. Let us therefore integrate over one of the azimuthal

angles. The angular distribution of the event z -axis is just the (θ, φ) -distribution given by

$$(2.4.8) \quad \frac{(2\pi)}{g_{11}^2 \sigma_{pt}} \frac{d^4 \sigma}{d\varphi \, d\cos\theta \, dx_1 \, dx_2} =$$

$$\frac{3}{8} (\sigma_U^{p.c.} + 2 \sigma_L^{p.c.}) \left\{ G_{A1} (1 + \alpha \cos^2 \theta) + \alpha G_{B1} \sin^2 \theta \cos 2\varphi \right. \\ \left. + \alpha G_{C1} \sin^2 \theta \sin 2\varphi + \epsilon G_{D4} \cos \theta \right\}$$

where

$$(2.4.9) \quad \alpha = \frac{\sigma_U^{p.c.} - 2 \sigma_L^{p.c.}}{\sigma_U^{p.c.} + 2 \sigma_L^{p.c.}}, \quad \epsilon = \frac{\sigma_U^{p.v.}}{\sigma_U^{p.c.} + 2 \sigma_L^{p.c.}}$$

This formula is valid in any order of α_s .

Perhaps more familiar is the (θ, N) -distribution resulting

from (2.4.4):

$$(2.4.10) \quad \frac{(2\pi)}{g_{11}^2 \sigma_{pt}} \frac{d^4 \sigma}{dN \, d\cos\theta \, dx_1 \, dx_2} = \frac{3(\sigma_U^{p.c.} + 2 \sigma_L^{p.c.})}{8} G_{A1}$$

$$\left\{ 1 + \alpha \cos^2 \theta + \alpha_t \sin^2 \theta \cos 2N + \alpha_i \sin 2\theta \cos N \right.$$

$$\left. + \frac{G_{A4}}{G_{A1}} \left[\alpha_9 \sin^2 \theta \sin 2N + \alpha_8 \sin 2\theta \sin N \right] + \epsilon_5 \frac{G_{D1}}{G_{A1}} \sin \theta \sin N \right\}$$

Here we have defined

$$(2.4.11) \quad \alpha_t = \frac{2 \sigma_T^{p.c.}}{\sigma_U^{p.c.} + 2 \sigma_L^{p.c.}}, \quad \alpha_i = -2\sqrt{2} \frac{\sigma_I^{p.c.}}{\sigma_U^{p.c.} + 2 \sigma_L^{p.c.}}$$

$$\alpha_9 = -2 \frac{\sigma_9^{p.v.}}{\sigma_U^{p.c.} + 2 \sigma_L^{p.c.}}, \quad \alpha_8 = 2\sqrt{2} \frac{\sigma_8^{p.v.}}{\sigma_U^{p.c.} + 2 \sigma_L^{p.c.}}$$

$$\epsilon' = -4\sqrt{2} \frac{\sigma_7^{p.v.}}{\sigma_U^{p.c.} + 2 \sigma_L^{p.c.}}, \quad \epsilon_5 = 4\sqrt{2} \frac{\sigma_5^{p.c.}}{\sigma_U^{p.c.} + 2 \sigma_L^{p.c.}}$$

Anticipating again the order α_s^2 result, the last line in (2.4.10) vanishes.

All angular distributions arrived at so far are valid in the

helicity frame. The cross sections σ_α correspond to coefficients of certain angular dependences and can therefore (in principle) be measured. But some coefficients appear only in double angle differential cross sections. However, by switching to the transversity frame certain linear combinations of these appear in single (polar) angle distributions. It is clear that this has experimental advantages. Using the relations (2.3.14) we can express the transversity distributions directly in terms of the helicity cross sections σ_α . As an example consider the $\bar{\theta}$ -distribution:

$$(2.4.12) \quad \frac{d^3 \sigma^f}{d\cos\bar{\theta} dx_1} = g_{11}^f \sigma_{pt} \left\{ \frac{3}{8} G_{A1} \left[(\bar{\sigma}_U + 2 \bar{\sigma}_L) + \cos^2 \bar{\theta} (\bar{\sigma}_U - 2 \bar{\sigma}_L) \right] + 2 \bar{\sigma}_L \cos \bar{\theta} G_{D4} \bar{\sigma}_6 \right\} \\ = g_{11}^f \sigma_{pt} \left\{ \frac{3}{8} G_{A1} \left[\left(1 + \frac{1}{2} \sin^2 \bar{\theta}\right) \sigma_U + 2 \left(1 - \frac{1}{2} \sin^2 \bar{\theta}\right) \sigma_L \right] + 2 \left(-1 + \frac{3}{2} \sin^2 \bar{\theta}\right) \sigma_T \right\} + \frac{3\sqrt{2}}{2} G_{D4} \cos \bar{\theta} \sigma_5$$

Since $\bar{\theta}$ is the angle between the normal and the e^- beam, the helicity cross section σ_5 (which should be zero up to order α_s^2) can be determined from the forward-backward asymmetry of the normal to the event plane, whereas a helicity-type analysis would require to measure a $\sin \theta \sin \bar{\theta}$ term (see (2.4.10)). In the transversity frame it is also much easier to check experimentally whether the $O(\alpha_s)$ relation $\sigma_2 = 2 \sigma_3$ or equivalently $\bar{\sigma}_1 = \bar{\sigma}_2$ is valid. These relations are true in order α_s but no longer in order α_s^2 (see section 7).

We observe (see (2.4.4)) that massless 3-parton events are in general described by 8 electro-weak couplings G_{Kr}

($K = A, B, C, D$; $r = 1, 4$), whereas up to and including the order α_s^2 merely 4 different couplings are necessary, namely G_{A1} , G_{B1} , G_{C1} and G_{D4} . Their explicit expressions are (2.4.13)

$$g_{11} G_{A1} = \left[\frac{2}{3} Q_e^2 Q_f^2 + 2 Q_e Q_f v_f \operatorname{Re} x_Z (v Z_1 - a Z_2) + (v_f^2 + a_f^2) |x_Z|^2 (v^2 + a^2) Z_1 - 2 v a Z_2 \right] \\ g_{11} G_{D4} = \left[2 Q_e Q_f a_f \operatorname{Re} x_Z (a Z_1 - v Z_2) + 2 v_f a_f |x_Z|^2 (2 v a Z_1 - (v^2 + a^2) Z_2) \right] \\ g_{11} G_{B1} = x_1 \left[Q_e^2 Q_f^2 + 2 Q_e Q_f v_f \operatorname{Re} x_Z + (v^2 - a^2) (v_f^2 + a_f^2) |x_Z|^2 \right] \\ - x_2 \left[2 Q_e Q_f a v_f \operatorname{Im} x_Z + 2 Q_e Q_f v_f \operatorname{Re} x_Z + (v^2 - a^2) (v_f^2 + a_f^2) |x_Z|^2 \right] \\ + x_1 \left[2 Q_e Q_f a v_f \operatorname{Im} x_Z \right]$$

These 4 coefficients, however, are the same as those in the 2-jet cross section /97/. Thus, compared with $q\bar{q}$ 2-jet events, no additional information regarding the Z^0 can be obtained from the study of 3-jet events. In particular, the (θ, φ) distribution reduces to the one for 2-jet events if we take $\alpha = 1$ and $\epsilon = 1$ in (2.4.8).

Note also, that the symmetric (p.c.) part of the cross section is (contrary to the statement of e.g. /35/) not the same as for the purely electro-magnetic annihilation. This is true only for the φ -integrated cross section (2.4.10) where the transverse beam polarization dependence drops out.

In the (θ, φ) distribution (2.4.8) the $(1 + \alpha \cos^2 \theta)$ -term of the normalized angular distribution does not feel the presence of the Z^0 . However, the Z^0 shows up in the φ -dependent terms with

strength G_{B1} / G_{A1} for $\cos 2\varphi$ and G_{C1} / G_{A1} for $\sin 2\varphi$. For naturally polarized beams these strength are given by $P^2 A_{21}$ and $P^2 A_{31}$ respectively. Their energy dependence is shown in Figs. 5 and 6.

The parity violating C-odd parts of the cross section are described up to order α_s^2 by merely two p.v. cross sections $\sigma_6^{p.v.}$ and $\sigma_7^{p.v.}$ and depend solely on one electro-weak, polarization dependent strength G_{D4} / G_{A1} (see (2.4.4,10)). It is usual to integrate the angle differential cross section over half spheres and then consider the asymmetries of the contributions in the two half spheres (see section 7.5) which clearly project out the p.v. helicity cross sections $\sigma_6^{p.v.}$ and $\sigma_7^{p.v.}$ in the numerator.

Both p.v. cross sections (and thereby the charge asymmetries defined in §7.5) are proportional to the strength

$$(2.4.14) \quad \frac{G_{D4}}{G_{A1}} = \frac{(h^- - h^+) g_{14} + (1 - h^- h^+) g_{44}}{(1 - h^- h^+) g_{11} + (h^- - h^+) g_{41}}$$

If the beam has no longitudinal polarization this strength is simply

$$(2.4.15) \quad \frac{G_{D4}}{G_{A1}} (h^- = h^+ = 0) = A_{44} = \frac{1}{g_{11}} \left\{ \text{Re} x_Z 2Q_e Q_f a_f + v 4a_f a_f |x_Z|^2 \right\}$$

with energy dependence shown in Fig. 7. Since the $(\gamma-Z)$ interference term (prop. $\text{Re } x_Z$) in A_{44} vanishes on top of the Z-resonance (see Fig. 7), we are left with the V-A interference term of the Z-coupling (prop. $v = -1 + 4\sin^2\theta_W$) which is small in the Weinberg-Salam theory for $\sin^2\theta_W$ close to 1/4. Longitudinal beam polarization is required in order to get noticeable contributions from the p.v. cross sections on top of the Z^0 . On the Z^0 the strength (2.2.14) is given by

$$(2.4.16) \quad \frac{G_{D4}}{G_{A1}} (s = m_Z^2) = h A_{14} (s = m_Z^2)$$

assuming, for the sake of simplicity, $\sin^2\theta_W = 1/4$. Here we have defined the polarization asymmetry h by

$$(2.4.17) \quad h \equiv \frac{h^- - h^+}{1 - h^- h^+}$$

We will discuss the strength G_{D4}/G_{A1} in §7.5 in the presentation of numerical results for the asymmetries.

So far we have looked at the p.c. and p.v. cross sections (structure functions) on and off the Z^0 solely in connection with their appearance in e^+e^- annihilation. Concerning Z's (and W's) produced at hadronic colliders the p.c. and p.v. contributions to the decays $Z \rightarrow q\bar{q}G$ and $W \rightarrow q'\bar{q}G$ are described by the same 4 p.c structure functions $H_1, \dots, H_4$ and the same 2 p.v. structure functions H_6 and H_7 as well as the vanishing structure functions $H_5^{p.c.}$, $H_8^{p.v.}$ and $H_9^{p.v.}$. Thus our order α_s^2 results apply also to the decay of hadronically produced Z's and W's. Depending on the polarization of the produced Z's and W's the 9 independent helicity cross sections can again be disentangled by angular distribution measurements as above. The polarization of hadronically produced W's and Z's at low and high p_T have been calculated to lowest order in /12,27,29,39,47/.

In the e^+e^- realm the p.c. and p.v. cross sections at LEP and SLC energies are of special interest, where the beam polarization and the huge statistics will enable to disentangle most of the cross sections. Of course, the order α_s^2 calculations have been done for massless quarks only, but this is certainly a good approximation for u, d, s, c and b quarks produced in Z decays or otherwise at LEP energies.

2.5. Final state polarization

In order to deal with final state polarization phenomena we have to generalize the helicity cross sections (2.2.18) to final state density matrices /63,85,87/. For the 3-parton final state the generalization consists in the replacement of the (final)

$$(2.5.1) \quad H_{\rho\sigma}^t \equiv \sum_{\lambda_1 \lambda_2 \lambda_3} \frac{(\mathbb{H}_{\rho\sigma}^t)_{\lambda_1 \lambda_2 \lambda_3; \lambda_1 \lambda_2 \lambda_3}}{(\mathbb{H}_{\rho\sigma}^t)_{\lambda_1 \lambda_2 \lambda_3; \lambda_1 \lambda_2 \lambda_3}} \rightarrow (\mathbb{H}_{\rho\sigma}^t)_{\lambda_1 \lambda_2 \lambda_3; \lambda_1' \lambda_2' \lambda_3'}$$

where λ_i denote the helicities of the final state quanta.

We will, however, limit our attention to few special cases relevant for experimental measurements: (single) quark and antiquark density matrices, and expressions for the angle-integrated, linear and circular polarization of the gluon where the polarization is specified relative to the event frame.

Quark Polarization

A single particle density matrix $(\mathbb{H}_{\rho\sigma}^t)_{\lambda\lambda'}$ such as the one for the quark is arrived at by summing over the helicities of the remaining particles in $(\mathbb{H}_{\rho\sigma}^t)_{\lambda_1 \lambda_2 \lambda_3; \lambda_1' \lambda_2' \lambda_3'}$ (2.5.1). As in the spin-averaged case where we defined nine $\sigma_\alpha^t \equiv \sigma_\alpha(\mathbb{H}^t)$ in terms of $\mathbb{H}_{\rho\sigma}^t$ (see (2.2.18)), we introduce nine combinations $(\Sigma_\alpha^t)_{\lambda\lambda'}$, $(\alpha=1, \dots, 9)$ of $(\mathbb{H}_{\rho\sigma}^t)_{\lambda\lambda'}$, e.g.

$$(2.5.2) \quad \mathbb{J}^{-1}(\Sigma_1^t)_{\lambda\lambda'} = (\mathbb{H}_{++}^t)_{\lambda\lambda'} + (\mathbb{H}_{--}^t)_{\lambda\lambda'}$$

$\sigma_\alpha(\mathbb{H}^t)$, the spin-averaged (hadronic) correlation cross sections from § 2.2, obey

$$\sigma_\alpha(\mathbb{H}^t) = \text{Tr} (\Sigma_\alpha^t)_{\lambda\lambda'} \equiv \sum_{\lambda} (\Sigma_\alpha^t)_{\lambda\lambda}$$

For a spin- $\frac{1}{2}$ particle (and for massless spin-1 particles) the

single particle density matrix $(\Sigma_\alpha^t)_{\lambda\lambda'}$ can be conventionally parametrized in terms of the unit and Pauli matrices τ_i as

$$(2.5.3) \quad (\Sigma_\alpha^t)_{\lambda\lambda'} = \frac{1}{2} \left\{ \sigma_\alpha(\mathbb{H}^t) \cdot 1 + \rho_{\alpha x}^t \tau_1 + \rho_{\alpha y}^t \tau_2 + \rho_{\alpha z}^t \tau_3 \right\}$$

The three component object $\vec{\rho}_\alpha^t$ is the (hadronic) correlation polarization vector for the quark and antiquark, respectively (or the correlation Stokes vector for the gluon). After contracting with the lepton tensors the $\vec{\rho}_\alpha^t$'s build up the (unnormalized) polarization (or Stokes) vector in much the same way as the $\sigma_\alpha(\mathbb{H}^t)$'s build up the helicity cross sections /63,85,87/.

The contraction with the lepton tensors can be done exactly in the same way as in the spin averaged case (see § 2.2). Also, since we are neglecting quark masses, the transverse components Σ_α^2 and Σ_α^3 vanish:

$$(2.5.4) \quad (\Sigma_\alpha^2)_{\lambda\lambda'} = (\Sigma_\alpha^3)_{\lambda\lambda'} = 0$$

We again identify

$$(2.5.5) \quad \Sigma_\alpha^{\text{p.c.}} \equiv \Sigma_\alpha^1 = \Sigma_\alpha^{VV}, \quad \Sigma_\alpha^{\text{p.v.}} \equiv \Sigma_\alpha^4 = \Sigma_\alpha^{VA}$$

For massless quarks one has a simple factorizing structure for the quark density matrix /63/

$$(2.5.6) \quad (\Sigma_\alpha^{\text{p.c.}})_{\lambda\lambda'} = \left[\frac{1}{2} (\sigma_\alpha^{\text{p.c.}} \cdot 1 + \vec{0} \cdot \vec{\tau}) \right]_{\lambda\lambda'} ; \quad \alpha = 1, 2, 3, 4, 5$$

$$\left[\frac{1}{2} (0 \cdot 1 + \sigma_\alpha^{\text{p.v.}} \cdot \tau_3) \right]_{\lambda\lambda'} ; \quad \alpha = 6, 7, 8, 9$$

$$(\Sigma_\alpha^{\text{p.v.}})_{\lambda\lambda'} = \left[\frac{1}{2} (0 \cdot 1 + \sigma_\alpha^{\text{p.c.}} \cdot \tau_3) \right]_{\lambda\lambda'} ; \quad \alpha = 1, 2, 3, 4, 5$$

$$\left[\frac{1}{2} (\sigma_\alpha^{\text{p.v.}} \cdot 1 + \vec{0} \cdot \vec{\tau}) \right]_{\lambda\lambda'} ; \quad \alpha = 6, 7, 8, 9$$

$$(\Sigma_\alpha^{\text{p.c.}/\text{p.v.}})_{\lambda\lambda'} = 0 \quad \text{in all other cases.}$$

Gluon Polarization

The gluon spin density matrix has more structure than the quark spin density matrix. Order α_s results can be found in /63,86,87/. We will restrict ourselves to the case of the angle-integrated gluon polarization, the most relevant one for experimental measurements. First of all we need the angle integrated cross section formula for the case where the gluon polarization has not been summed up (i.e. where we did not use the simplifications valid in the spin-averaged case). We get ($\zeta = +, -, ||, \perp$)

$$(2.5.8) \quad d\sigma_\zeta = \frac{1}{\sigma_{pt}} \frac{d^2\sigma_\zeta}{dx_1 dx_2} = \mathcal{G}_{41} \frac{-1}{64 \pi^2} \left\{ G_{A1} (H^\mu p \cdot c)_\mu \zeta + G_{A4} (H^\mu p \cdot v)_\mu \zeta \right\}$$

where

$$(2.5.9) \quad (H^\mu)_\mu \zeta = \epsilon^\zeta_\alpha H^{\alpha\beta}_{\mu\nu} \epsilon^{\zeta*}_{\beta} g^{\mu\nu}$$

and $H^{\alpha\beta}_{\mu\nu}$ is the density matrix of gluon. In terms of the amplitude $T_{\mu\alpha}$ defined in (2.1.14) it becomes

$$(2.5.10) \quad H^{\alpha\beta}_{\mu\nu} = \text{Tr } \gamma q_1 T^\alpha_\mu \gamma q_2 T^\beta_\nu$$

Here ϵ^ζ_α are the polarization vectors of the gluon

$$(2.5.11) \quad \begin{aligned} \epsilon^\parallel_\beta &= -\frac{N}{2} (t_{23} q_{1\beta} - t_{13} q_{2\beta}) \\ \epsilon^\perp_\beta &= N \epsilon(\beta, q_2, q_1, q_3) \\ \epsilon^\pm_\beta &= \frac{-1}{\sqrt{2}} (\pm \epsilon^\parallel_\beta + i \epsilon^\perp_\beta) \end{aligned}$$

and N is a normalization constant ($t_{ij} \equiv 2q_i q_j$)

$$(2.5.12) \quad N = \frac{1}{\sqrt{t_{12} t_{13} t_{23}}}$$

With these definitions we can write down the gluon's linear polarization (relative to the event plane)

$$(2.5.13) \quad P_{lin} = \frac{d\sigma_{||} - d\sigma_{\perp}}{d\sigma_{||} + d\sigma_{\perp}}$$

We find /63/ that the p.c. part of the quark density matrices are unpolarized for (the symmetric combinations) $\alpha = 1, 2, 3, 4, 5$ and longitudinally polarized for the (antisymmetric combinations) $\alpha = 6, \dots, 9$ and vice versa for the p.v. part. This complementary behaviour is a consequence of the masslessness and planarity of the 3-jet events. Notice also that the quark- density matrices are already given by the helicity cross sections σ_α , i.e. they do not contain any further information.

Neglecting the quark masses the quark and antiquark

helicities are correlated: $\lambda_q = -\lambda_{\bar{q}}$. Thus the antiquark correlation density matrices $(Z^t_{\alpha\lambda\lambda'})_{\lambda\lambda'}$ can be obtained from

$$(2.5.6) \text{ by simply replacing } \tau_3 \rightarrow -\tau_3.$$

We will not discuss the general differential cross section distribution obtained by inserting (2.5.6) in (2.2.29) but consider here as an example the polar angle distribution of the quark density matrix. We obtain for the spin density matrix of the quark (G_{Kr} is defined in (2.4.5))

$$(2.5.7) \quad \frac{2}{\sigma_{pt}} \frac{d\rho_{\lambda\lambda'}^q}{dx_1 dx_2 d\cos\theta} = \mathcal{G}_{41} (G_{A1} \cdot 1 + G_{A4} \cdot \tau_3) \left[\frac{3}{8} (1 + \cos^2\theta) \sigma_U^{p.c.} + \frac{3}{4} \sin^2\theta \sigma_L^{p.c.} \right] + (G_{D4} \cdot 1 + G_{D1} \cdot \tau_3) \frac{3}{4} \cos\theta \sigma_6^{p.v.}$$

The degree of longitudinal polarization p^q_2 of the quark is obtained from (2.5.7) by calculating the ratio $\text{Tr}(\rho \tau_3) / \text{Tr } \rho$. Proposals for the measurement of the quark's longitudinal polarization exist /67/, although experimental tests have not been done so far.

and it's circular polarization

$$(2.5.14) \quad P_{\text{circ}} = \frac{d\sigma_+ - d\sigma_-}{d\sigma_+ + d\sigma_-}$$

A discussion of these polarizations and order α_s^2 results to them will be presented in §7.6.

3. JET CROSS SECTIONS

3.1 Jet definitions and dressing schemes

In this section we describe how to define a physically finite 3-jet cross section. The $O(\alpha_s^2)$ 3-parton and 4-parton cross sections¹ are individually IR/M (infrared and mass) divergent but the divergences cancel in the sum (KLN theorem). However, the remaining contributions depend on the precise definition of the 3-jet region /52/. Therefore the definition of the 3-jet region has to be carefully specified.

The calculation of the $O(\alpha_s^2)$ "dressed" (i.e. finite) 3-jet cross section can be described in three steps /51/:

- (i) A criterion is introduced to define resolvable 4-jet events
- (ii) The 4-parton cross section is integrated over unresolved configurations

- (iii) These integrated, unresolved contributions are interpreted as corrections to the 3-jet cross section:

$$(3.1.1) \quad d\sigma[3\text{-jets}] = d\sigma^V[3\text{-partons}] + \int \frac{d\sigma^R[4\text{-partons}]}{\text{unresolved pairs}}$$

The addition of the $O(\alpha_s^2)$ virtual (V) corrections to the real (R) contributions integrated over the 3-jet domains of phase space leads to the cancellation of the IR/M divergences a la KLN. The 3-jet cross section (3.1.1) is finite, but resolution dependent.

We define resolvable parton pairs in terms of a mass cut

$$(3.1.2) \quad y_{ij} = \frac{(p_i + p_j)^2}{q^2} > y$$

¹ $\alpha_s = g^2/(4\pi)$ where g is the strong coupling constant.

Either scheme is suitable for evaluating $O(\alpha_s^2)$ cross sections, provided that (1) the same scheme is used to evaluate both the 3-jet and the 4-jet contributions, and (2) the hadronization scheme used in the full Monte Carlo calculations is continuous in its treatment of soft and/or collinear partons /101/. In order to pinpoint the difference between the two DD and PFD dressing schemes we exhibit this difference explicitly in the simpler case of $O(\alpha_s)$ 2-jet production in the next section.

In the $O(y^0)$ approximation we found the DD scheme to be more convenient for the evaluation of all helicity cross sections.

Of course, our results are valid only up to higher order y corrections. As the analysis for the angle integrated cross section showed /53/, these higher order y corrections (for this case) are small for well defined 3-jet events ($T < 0.85$), but could become large in the 2-jet region for $T > 0.95$ depending on the cut-off y . If a higher accuracy for a given cutoff y_1 is desired one can always choose first a smaller y_2 ($y_2 < y_1$) and correct for the difference numerically by adding the 4-parton contribution in the desired $(y_2 - y_1)$ -bin with two partons lying in this bin averaged over.

where y is the cut-off parameter. Unfortunately, the unresolved phase space regions of the 4-parton process are not uniquely determined. Different identifications of certain finite 4-parton configurations as being either 3-jet or 4-jet contributions lead to different dressing schemes. Two schemes have proved useful in the calculation of the angle-integrated $O(\alpha_s^2)$ 3-jet cross section /75,54,50,51/:

1. Direct dressing scheme (DD) (which will be used by us):

Resolvable 4-jet configurations are defined by

$$(3.1.3) \quad d\sigma[4\text{-jets}; DD] \equiv d\sigma[4\text{-partons}] \prod_{ij} \theta(y_{ij} - y)$$

The θ -functions specifically exclude phase space regions in which any of the parton pair masses is inside the resolvable cut-off.

2. Partial fractioned dressing (PFD):

By repeated partial fractioning, the 4-parton squared matrix

element is first re-written in the form

$$(3.1.4) \quad |M(4\text{-partons})|^2 = \sum_{i,j} \frac{F_{ij}(y_{mn})}{y_{ij}}$$

where, for $y_{ij} \neq 0$, the coefficient functions F_{ij} are finite for all possible values of the other pair masses. The resolved 4-jet cross section in PFD is²

$$(3.1.5) \quad d\sigma[4\text{-jets}; PFD] \equiv \left\{ \sum_{i,j} \frac{F_{ij}}{y_{ij}} \theta(y_{ij} - y) \right\} d\text{ps}^{(4)}$$

We note that the θ -functions in (3.1.5) exclude only configurations in which the appropriate singular pair mass is below the cutoff. Contrary to the DD scheme, the PFD scheme includes some non-singular soft parton configurations.

² $d\text{ps}^{(4)}$ is the 4-particle phase space

3.2 An illustrative example: $O(\alpha_s)$ corrections to 2-jet production

The purpose of this section is twofold: We want to demonstrate the scheme- dependence of a dressed jet cross section, and to introduce some technical notions concerning the phase space integration of the tree graph hadron tensors. To this end we consider the 2-jet reaction¹ $e^+e^- \rightarrow \gamma \rightarrow q\bar{q}$ to $O(\alpha_s)$ /52/. The generalization to arbitrary final states is obvious. Remarks concerning the $O(\alpha_s^2)$ 3-jet cross section will be given at the end of this section.

The 2-jet cross section is given by

$$(3.2.1) \quad d\sigma[2\text{-jet}; \alpha_s] = d\sigma^V[2\text{-partons}; \alpha_s] + \int d\sigma^R[3\text{-partons}; \alpha_s]_{\text{unresolved pairs}}$$

The Feynman diagrams for the virtual (2-parton final state) and real (3-parton final state) contributions are shown in Fig. 8. We denote the 3-parton momenta by q_i ($i=1,2,3$) and the 2-parton ones by r_j ($j=1,2$) with $q_i^2 = r_j^2 = 0$. In order to regularize the singular integrals occurring in each part we use dimensional regularization. We work in $n = 4-2\epsilon$ dimensions and take final state partons to be on-shell and massless. The advantage of a covariant formulation of the hadron tensors is apparent since it can easily be continued from 4 to $n \neq 4$ dimensions (for p.v. difficulties see below).

However, we must first specify the precise manner in which the integrated, irresolvable 3-parton configurations are identified with 2-parton states. We define the identification in

¹The additional difficulties that arise in the p.v. case will be dealt with later on.

terms of jet energies. Irresolvable 3-parton configurations are identified with states of 2 massless partons of the same (total) energy. In Fig. 8b, for example, the irresolvable quark + gluon pair is to be treated as a single massless quark of the same total energy.

For the explicit 2-jet cross section formula we have²

$$(3.2.2) \quad d\sigma[2\text{-jet}; \alpha_s] = K \left\{ L_{\mu\nu} H^{\mu\nu}[2\text{-partons}; \alpha_s] d\text{ps}^{(2)} + \int_{\text{u.pairs}} L_{\mu\nu} H^{\mu\nu}[3\text{-partons}; \alpha_s] d\text{ps}^{(3)} \right\}$$

For the 3-parton phase space (massless) we obtain

$$(3.2.3) \quad d\text{ps}^{(3)} = \prod_{i=1}^3 \frac{d^{n-1}q_i}{2E_i(2\pi)^{n-1}} (2\pi)^n \delta(q-q_1-q_2-q_3) \\ = d\text{ps}^{(2)} \int dv f(z_{23}, v)$$

Note that the 3-parton phase space has been recast into a form where the 2-massless parton phase space $d\text{ps}^{(2)}$ factorizes in order to facilitate the summation with the virtual contributions in (3.2.2). In (3.2.3) we have defined

$$(3.2.4) \quad f(z_{23}, v) = \frac{q^2}{16\pi^2} \left[\frac{4\pi\mu^2}{q^2} \right]^{\epsilon} \frac{1}{F(1-\epsilon)} z_{23}^{-\epsilon} (1-z_{23})^{1-2\epsilon} v^{-\epsilon} (1-v)^{-\epsilon}$$

Here $z_{23} = 2q_2 q_3 / q^2$ and $v = z_{13} / (1-z_{23}) = \frac{1}{2}(1-\cos\theta)$, where θ is the polar angle between the gluon (q_3) and the quark (q_1). μ is an arbitrary scale parameter. The integration limits will be specified later on.

From

$$(3.2.5) \quad d\sigma[2\text{-jet}; \alpha_s] \equiv K L_{\mu\nu} d\text{ps}^{(2)} H_{\mu\nu}[2\text{-jet}; \alpha_s]$$

we define the (finite) 2-jet hadron tensor

²In the pure electro-magnetic case $K = e^4/(2q^2)$.

$$(3.2.6) \quad \begin{aligned} H_{\mu\nu}^{(2)}(r_1, r_2) &\equiv H_{\mu\nu}[2\text{-jet}; \alpha_s] \\ &= H_{\mu\nu}^{(2)}(r_1, r_2)^{\text{Virt}} + H_{\mu\nu}^{(2)}(r_1, r_2)^{\text{Real}} \end{aligned}$$

Here we have identified

$$(3.2.7) \quad H_{\mu\nu}^{(2)}(r_1, r_2)^{\text{Virt}} \equiv H_{\mu\nu}[2\text{-partons}; \alpha_s]$$

and

$$(3.2.8) \quad H_{\mu\nu}^{(2)}(r_1, r_2)^{\text{Real}} = \int dz_{23} dv f(z_{23}, v) B_{\mu\nu}^{(3)}(q_1, q_2, q_3)$$

with the 3-parton (Born) hadron tensor

$$(3.2.9) \quad B_{\mu\nu}^{(3)}(q_1, q_2, q_3) \equiv H_{\mu\nu}[3\text{-partons}; \alpha_s]$$

The final state 3-parton phase space is

$$(3.2.10) \quad 0 \leq z_{23} \leq 1, \quad 0 \leq z_{13} \leq 1 - z_{23}$$

Using the direct dressing scheme the regions

$$(3.2.11) \quad \begin{aligned} \text{(i)} \quad & 0 \leq z_{23} \leq y, \quad 0 \leq z_{13} \leq y \\ \text{(ii)} \quad & 0 \leq z_{23} \leq y, \quad y \leq z_{13} \leq 1 - z_{23} \\ \text{(iii)} \quad & y \leq z_{23} \leq 1 - z_{13}, \quad 0 \leq z_{13} \leq y \end{aligned}$$

define the phase space for unresolved emissions, as indicated by

the shaded areas in Fig. 9a. This fixes the integration limits for the real hadron tensor

$$(3.2.12) \quad H_{\mu\nu}^{(2)}(r_1, r_2)^{\text{Real}} = \int_0^y dz_{23} \int_0^{y/(1-z_{23})} dv f(z_{23}, v) B_{\mu\nu}^{(3)}(q_1, q_2, q_3)$$

Here we have used the symmetry of the 3-parton hadron tensor $B_{\mu\nu}^{(3)}$ under $z_{23} \leftrightarrow z_{13}$ with the appropriate proviso concerning double counting of region (i). Clearly, the integrated 3-parton hadron tensor must have the 2-parton (Born) structure, i.e.

$$(3.2.13) \quad \begin{aligned} H_{\mu\nu}^{(2)}(r_1, r_2)^{\text{Real}} &\stackrel{!}{=} H_{\mu\nu}^{\text{R}} B_{\mu\nu}^{(2)}(r_1, r_2) \\ &\equiv H_{\mu\nu}^{\text{R}} H_{\mu\nu}[2\text{-partons}; \alpha_s^0] \end{aligned}$$

with some structure function H^{R} and the 2-parton Born hadron

tensor

$$(3.2.14) \quad B_{\mu\nu}^{(2)}(r_1, r_2) = 4(r_1^{\mu} r_{2\nu} + r_2^{\mu} r_{1\nu} - \frac{q^2}{2} g_{\mu\nu}).$$

Contracting both sides of (3.2.13) with the (n-dimensional) metric tensor $g_{\mu\nu}$ allows one to determine H^{R} . A straightforward calculation gives¹

$$(3.2.15) \quad \begin{aligned} g^{\mu\nu} B_{\mu\nu}^{(3)}(q_1, q_2, q_3) &= 8g^2 N_{\text{CF}} \\ &\left\{ \frac{z_{23}}{z_{13}} + \frac{z_{13}}{z_{23}} + 2 \frac{(1 - z_{23} - z_{13})}{z_{13} z_{23}} - \epsilon \left[\frac{z_{23}}{z_{13}} + \frac{z_{13}}{z_{23}} + 2 \right] \right\} \end{aligned}$$

Restricting ourselves to terms of order y^0 we may replace

$$(3.2.16) \quad \begin{aligned} g^{\mu\nu} B_{\mu\nu}^{(3)} &\rightarrow \frac{1}{z_{23}} \lim_{z_{23} \rightarrow 0} (z_{23} g^{\mu\nu} B_{\mu\nu}^{(3)}) \\ &= 4g^2 N_{\text{CF}} \frac{2}{z_{23}} \frac{1}{z_{23}} \left[(1 - \epsilon)v + 2 \frac{1 - v}{v} \right] \end{aligned}$$

After doing the integrations (3.2.12) we obtain

$$(3.2.17) \quad \begin{aligned} H_{\mu\nu}^{(2)}(r_1, r_2)^{\text{Real}} &= B_{\mu\nu}^{(2)}(r_1, r_2) g_{\text{NCF}}^2 C \\ &\left\{ \frac{2}{\epsilon} + \frac{3}{\epsilon} + 7 - 2 \ln^2 y - 3 \ln y + \frac{\pi^2}{3} \right\} \end{aligned}$$

where

$$(3.2.18) \quad C = \frac{1}{8\pi^2} \frac{r(1 - \epsilon)}{r(1 - 2\epsilon)} \left[\frac{4\pi\mu^2}{q^2} \right]^\epsilon \left(1 - \frac{\pi^2}{3} \epsilon^2 \right)$$

The virtual $O(\alpha_s)$ contributions (loop graph contributions)

$H_{\mu\nu}^{(2)}(r_1, r_2)^{\text{Virt}}$ can be evaluated by using standard n-dimensional loop integrals as e.g. in /70/. They are proportional to the Born 2-parton hadron tensor. A straightforward calculation gives

$$(3.2.19) \quad H_{\mu\nu}^{(2)}(r_1, r_2)^{\text{Virt}} = B_{\mu\nu}^{(2)}(r_1, r_2) C g_{\text{NCF}}^2 \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 \right\}$$

Finally, adding up the two (real plus virtual) contributions (3.2.17) and (3.2.19) we obtain the $O(y^0)$ 2-jet hadron tensor in

¹ N_{C} is the number of colours and $C_{\text{F}} \equiv (N_{\text{C}}^2 - 1)/(2N_{\text{C}})$.

DD scheme

$$(3.2.20) \quad H_{\mu\nu}[2\text{-jet}; \alpha_s]^{DD} = H_{\mu\nu}[2\text{-partons}; \alpha_s^0] \\ \frac{\alpha_s}{2\pi} N_{CF}^C (-1 - 2\ln^2 y - 3\ln y + \frac{\pi^2}{3})$$

In the PFD scheme, the poles in (3.2.15) are first separated

by writing

$$(3.2.21) \quad g^{\mu\nu} B_{\mu\nu}^{(3)}(q_1, q_2, q_3) = 8g^2 K C_F \\ \left\{ \frac{1}{z_{23}} \left[z_{13}(1-\epsilon) - 2 + \frac{2}{z_{23} + z_{13}} - \epsilon z_{23} \right] \right\} + \left\{ z_{23} \leftrightarrow z_{13} \right\}$$

The unresolved phase space for the first term is the region

$$(3.2.22) \quad 0 \leq z_{23} \leq y, \quad 0 \leq z_{13} \leq 1 - z_{23}$$

indicated by the shaded area in Fig. 9b. Integrating (3.2.21) over

unresolved regions and adding the virtual terms in (3.2.19) gives

$$(3.2.23) \quad H_{\mu\nu}[2\text{-jet}; \alpha_s]^{PFD} = H_{\mu\nu}[2\text{-partons}; \alpha_s^0] \\ \frac{\alpha_s}{2\pi} N_{CF}^C (-1 - 2\ln^2 y - 3\ln y)$$

The difference between (3.2.20) and (3.2.23) is the shift of

a contribution

$$(3.2.24) \quad \Delta H = \frac{\alpha_s}{2\pi} N_{CF}^C \frac{\pi^2}{3}$$

from the 2-jet cross section of the DD scheme to the 2-jet cross

section of the PFD scheme. This term comes from the double-

denominator integral

$$(3.2.25) \quad \Delta H = 2 Cg^2 N_{CF}^C \int_0^1 \frac{dz_{23}}{z_{23}} \int_0^{\min(y, 1-z_{23})} \frac{2dz_{13}}{z_{13} + z_{23}} \\ = 4 N_{CF}^C \frac{\alpha_s}{2\pi} \int_0^1 \frac{dz_{23}}{z_{23}} \ln \left[1 + \frac{y}{z_{23}} \right] + O(y)$$

Note that ΔH is a constant for $y \rightarrow 0$, even though the integrand vanishes for $y \rightarrow 0$. Similar differences occur between the two

dressing schemes for the higher order y corrections.

With regard to the $O(\alpha_s^2)$ dressing of 3-jets there is a similar $O(y^0)$ difference between the two dressing schemes. The $O(y^0)$ difference results from the integral $(Y_{134} \equiv Y_{13} + Y_{14} + Y_{34})$

$$(3.2.26) \quad H \equiv \int [dps]^{(4)} \theta(Y_{13} - y) \theta(y - Y_{14}) \left[\frac{Y_{134}}{Y_{13}(Y_{13} + Y_{14})} \right]$$

which is attributed to the 3-jet cross section in the DD scheme but to the 4-jet cross section in the PFD scheme. We want to stress again that the difference between the two dressing schemes survives in the limit $y \rightarrow 0$.

As shown in references /52/ for the angle-integrated 3-jet cross section, the differences in the results of DD and PFD are precisely due to different interpretations of finite terms such as H. There is no obvious "correct" way to include subleading terms in a dressed jet calculation. What matters is simply that the 3-jet and 4-jet terms be treated in a consistent manner. We found that the DD scheme is more convenient for doing the $O(y^0)$ calculation /64,73/. Further, the analytic expressions of the PFD scheme do depend on specifics of how the partial fractioning is performed. However, the authors of /51/ claim that the numerical results for the 3-jet cross sections are largely unaffected by partial fractioning ambiguities.

However, it should be noted that, for non-zero y , subleading $y \ln y$ -terms in the DD and PFD calculation of the angle-integrated cross section were found to be different /52/. The reason of this persistent disagreement is still unknown /51/. The subleading differences are probably non-trivial for $y \geq 0.01$.

An estimate of the validity of the $O(y^0)$ approximation for given values of cut-off y and thrust T can be obtained from the angle-integrated cross section of /51,53/. For a y -range $y = 0.01-0.04$ the corrections to the angle-integrated cross section are small ($\leq 4\%$) for well defined 3-jet events, i.e. for $T \leq 0.85$, while they become large (up to 30%) in the 2-jet region, i.e. for $T \geq 0.95$.

4. INTEGRATION OF THE P.C. 4-PARTON CROSS SECTIONS

4.1 Defining the task

We start this section by describing how the p.c. perturbative 3-jet corrections to $O(\alpha_s^2)$ are obtained. To this end it is sufficient to consider the pure vector-current vector-current hadron tensor $H_{\mu\nu}^{VV}$ describing

$$\text{vector-current}(q) \rightarrow q(q_1)\bar{q}(q_2)G(q_3).$$

The axial vector current-axial vector current hadron tensor $H_{\mu\nu}^{AA}$ is obtained by the chiral invariance relation $H_{\mu\nu}^{AA} = H_{\mu\nu}^{VV}$ valid in massless QCD. This relation holds true regardless of the subtleties of treating γ_5 within dimensional regularization (see Appendix A). Due to the equality of $H_{\mu\nu}^{VV}$ and $H_{\mu\nu}^{AA}$ the relations (see (2.3.2))

$$(4.1.1) \quad H_{\mu\nu}^{p.c.} \equiv H_{\mu\nu}^1 \equiv \frac{1}{2}(H_{\mu\nu}^{VV} + H_{\mu\nu}^{AA}) = H_{\mu\nu}^{VV}$$

and

$$(4.1.2) \quad H_{\mu\nu}^2 \equiv \frac{1}{2}(H_{\mu\nu}^{VV} - H_{\mu\nu}^{AA}) = 0$$

remain unchanged even in $n \neq 4$ dimensions. (In the following we therefore can drop the index "1" = p.c.).

The 3-jet hadron tensor in $O(\alpha_s^2)$ is defined in complete analogy to the 2-jet hadron tensor in $O(\alpha_s)$ discussed in the previous section:

$$(4.1.3) \quad H_{\mu\nu}[3\text{-jet}; \alpha_s^2] \equiv H_{\mu\nu}^{(3)}(q_1, q_2, q_3; y) \\ = H_{\mu\nu}^{(3)}(q_1, q_2, q_3)_{\text{Virt}} + H_{\mu\nu}^{(3)}(q_1, q_2, q_3; y)_{\text{Real}}$$

where

$$(4.1.4) \quad H_{\mu\nu}^{(3)}(q_1, q_2, q_3)_{\text{Virt}} \equiv H_{\mu\nu}[3\text{-partons}; \alpha_s^2]$$

and (u.pairs $\equiv$ unresolved pairs)

$$(4.1.5) \quad H_{\mu\nu}^{(3)}(q_1, q_2, q_3; Y)_{\text{Real}} = \int dR H_{\mu\nu} [4\text{-partons}; \alpha_s^2]_{u.\text{pairs}}$$

Here dR denotes the remainder of the splitting of the (Lorentz invariant) 4-particle (massless) phase space in the one for 3 (massless!) particles times a rest:

$$(4.1.6) \quad dPS^{(4)} = dPS^{(3)} dR$$

We will specify the decomposition (4.1.6) in the next section when discussing the 4-parton kinematics. Let us here only add one comment concerning (4.1.6). The analogous decomposition (3.2.3) of the 3-parton phase space is exact to any order in Y . However, when trying to regroup the $(3n-4)$ -fold differential 4-particle phase space $d^{3n-4}PS^{(4)}$ (n is the space-time dimension) according to (4.1.6), we have to work out the Euler angles connecting the beam and the event frame in n dimensions. In order to avoid this complexity we do not integrate the product $L^{\mu\nu} H_{\mu\nu}$ depending on these Euler angles, but rather consider a full set of scalars built up from the hadron tensor which do not depend on these Euler angles. Doing so, we can integrate out all relative angles in (4.1.6).

A full set of scalars defining the hadron tensor and which do not depend on relative angles is arrived at by defining the following five projectors $P_{\mu\nu}^i$ built from the 3-jet momenta q_1, q_2, q_3 and the $(n\text{-dimensional})$ metric tensor $g_{\mu\nu}$ ($E_i = q_{i0}$)

$$(4.1.7) \quad P_{\mu\nu}^1 = -g_{\mu\nu}, \quad P_{\mu\nu}^2 = \frac{q_{1\mu} q_{1\nu}}{E_1^2}, \quad P_{\mu\nu}^3 = \frac{q_{2\mu} q_{2\nu}}{E_2^2}$$

$$P_{\mu\nu}^4 = \frac{q_{3\mu} q_{3\nu}}{E_3^2}, \quad P_{\mu\nu}^5 = \frac{q_{1\mu} q_{2\nu} - q_{2\mu} q_{1\nu}}{E_1 E_2}$$

With these projectors we construct the scalars (or projections of the hadron tensor) $S^{(i)}$

$$(4.1.8) \quad S^{(i)} = p_{\mu\nu}^i H^{\mu\nu} [3\text{-jet}; \alpha_s^2]$$

Then each helicity cross sections σ_α ($\alpha = U, L, T, I, 5$) is a linear combination of the $S^{(i)}$. In the next sections we will show how to obtain the real (tree) graph contributions to the scalars $S^{(i)}$. The corresponding virtual contributions will be evaluated in section 6.

For later reference we quote here the scalars built from the lowest order 3-parton hadron tensor $B_{\mu\nu}^{(3)P.C.}(q_1, q_2, q_3) \equiv H_{\mu\nu}^{P.C.}[3\text{-parton}; \alpha_s] / 70/:$

$$(4.1.9a) \quad S_{\text{Born}(13, z_{23})}^{(i)} \equiv p_{\mu\nu}^i B_{\mu\nu}^{(3)P.C.}(q_1, q_2, q_3)$$

where

$$(4.1.9b) \quad S_{\text{Born}(13, z_{23})}^{(1)} = (1-\epsilon) \left[\frac{z_{13}}{z_{23}} + \frac{z_{23}}{z_{13}} \right] + \frac{2z_{12}}{z_{13}^2 z_{23}} - 2\epsilon$$

$$S_{\text{Born}(13, z_{23})}^{(2)} = 2(1-\epsilon) \frac{z_{12}}{(1-z_{23})^2}$$

$$S_{\text{Born}(13, z_{23})}^{(3)} = S_{\text{Born}(23, z_{13})}^{(2)}$$

$$S_{\text{Born}(13, z_{23})}^{(4)} = 4 \frac{z_{12}}{(1-z_{12})^2}$$

$$S_{\text{Born}(13, z_{23})}^{(5)} = 0$$

Note that for convenience we have defined $S_{\text{Born}}^{(i)}$ without the factor $8g^2 N_C^2 C_F$.

We give also explicit expressions for the lowest order p.c. structure functions. Referring to (2.3.9) we write

$$(4.1.10) \quad 4g^2 N_C^2 C_F B_{\mu\nu}^{(3)}(q_1, q_2, q_3) = \sum_{k=1}^5 H_i^B \frac{H_i^1}{\mu\nu}$$

The Born structure functions are given by /65/

$$(4.1.11) \quad H_i^B = 4g^2 N_C^2 C_F H_i^B$$

¹Explicit relations will be given in section 7.2.

where

$$\begin{aligned}
 (4.1.12) \quad H_1^B &= - (B_1 - \epsilon B_3) = - S_{\text{Born}}^{(1)}(z_{13}, z_{23}) \\
 H_2^B &= H_3^B = - (1-\epsilon) B_2 \\
 H_4^B &= \epsilon B_2 \\
 H_5^B &= 0
 \end{aligned}$$

Here we have defined $(x_i \equiv 1 - z_{jk}; i, j, k \text{ cyclic})$

$$\begin{aligned}
 (4.1.13) \quad B_1 &= \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \\
 B_2 &= \frac{x_3^2}{(1-x_1)(1-x_2)} \\
 B_3 &= \frac{x_3^2}{(1-x_1)(1-x_2)}
 \end{aligned}$$

4.2 Kinematics

Our aim is to integrate the 4-parton processes $e^+e^- \rightarrow q\bar{q}GG$ and $e^+e^- \rightarrow q\bar{q}q\bar{q}$ over the unresolved phase space regions, i.e. over the various 2-parton sub-phase spaces up to an invariant mass cut-off y . We generally follow the notation of /ERT,42/ and denote the momenta occurring in the 4-parton $\rightarrow$ 3-jet reduction by $(p_1, p_2, p_3, p_4) \rightarrow (q_1, q_2, q_3)$, where the p_i are n-dimensional and the q_i are 4-dimensional.

The kinematics of the 3-jet process

$$(4.2.1) \quad e^+e^- \rightarrow q(q_1) + \bar{q}(q_2) + G(q_3)$$

are specified by the variables¹

$$(4.2.2) \quad z_{ij} \equiv \frac{(q_i + q_j)^2}{q^2} = \frac{t_{ij}}{q^2} \quad (i \neq j)$$

which satisfy the energy conservation constraint

$$(4.2.3) \quad z_{12} + z_{13} + z_{23} = 1$$

They are related to the scaled energies (in the cms $\vec{q}=0$) x_i of the partons by (i, j, k cyclic)

$$(4.2.4) \quad x_i \equiv \frac{2E_i}{\sqrt{s}} = 1 - z_{jk}$$

Consider next the 4-parton process

$$(4.2.5) \quad e^+e^- \rightarrow q(p_1) + \bar{q}(p_2) + G(p_3) + G(p_4)$$

in which one of the pair masses is irresolvably small. The corresponding kinematical variables are

$$(4.2.6) \quad y_{ij} \equiv \frac{(p_i + p_j)^2}{q^2} = \frac{s_{ij}}{q^2}, \quad \sum_{i \neq j} y_{ij} = 1$$

¹ $q \equiv q_1 + q_2 + q_3$

Since we are interested in calculating a Lorentz invariant quantity, we are at liberty to evaluate different terms in the transition probability in different Lorentz frames. For example, if we are interested in the case where

$$(4.2.7) \quad Y_{13} \leq Y$$

for some resolvability cut-off Y , we work in the "1-3 system", i.e. the 1-3 cm system. In the limit (4.2.7), the 4-parton system is to be interpreted as a dressed 3-jet configurations with jet momenta

$$(4.2.8) \quad q_1 = p_1 + p_3, \quad q_2 = p_2, \quad q_3 = p_4$$

Inserting these identifications in (4.2.2) gives[†]

$$(4.2.9) \quad z_{12} = Y_{123}, \quad z_{13} = Y_{134}, \quad z_{23} = Y_{24} - Y_{13}$$

where we have introduced the abbreviation $(i \neq j \neq k \neq i)$

$$(4.2.10) \quad \frac{s_{ijk}}{q^2} \equiv Y_{ijk} \equiv Y_{ij} + Y_{ik} + Y_{jk}$$

Restrictions on allowed values of Y_{ij} are discussed below.

In addition to z_{ij} and Y_{13} , (n-2) angular variables are needed to completely specify the 4-parton kinematics. There is some freedom in selecting these variables. It is convenient to use the "standard" or "ERT/42/" frame², i.e. the c.m. frame of the composite $p_1 + p_3$ defined by

(4.2.11)

$$p_1 = \frac{\sqrt{s_{13}}}{2} (1, +\sin\theta \sin\theta_1, \dots, \sin\theta_{n-3}, \dots$$

$$\dots, +\sin\theta \sin\theta_1 \cos\theta_2, +\sin\theta \cos\theta_1, +\cos\theta)$$

$$p_3 = \frac{\sqrt{s_{13}}}{2} (1, -\sin\theta \sin\theta_1, \dots, \sin\theta_{n-3}, \dots$$

$$\dots, -\sin\theta \sin\theta_1 \cos\theta_2, -\sin\theta \cos\theta_1, -\cos\theta)$$

$$p_2 = \frac{s_{123} - s_{13}}{2\sqrt{s_{13}}} (1, 0, \dots, 0, 1)$$

$$p_4 = \frac{s_{134} - s_{13}}{2\sqrt{s_{13}}} (1, 0, \dots, 0, \sin\beta, \cos\beta)$$

The angles θ and θ_i ($i=1, \dots, n-3$) are the independent variables required to complete the kinematics whereas 4-momentum conservation constrains β :

$$(4.2.12) \quad \frac{1}{2}(1 - \cos\beta) \equiv \lambda = \frac{Y_{13}(z_{23} + Y_{13})}{(z_{13} - Y_{13})(z_{12} - Y_{13})}$$

The requirement $\cos\beta > -1$ then yields the constraint

$$(4.2.13) \quad Y_{13} < z_{12} z_{13}$$

The 3-jet identification in (4.2.8) is kinematically consistent only if (4.2.13) is satisfied. Since we are calculating to $O(Y^0)$ only, Y can always be chosen sufficiently small to satisfy (4.2.13) or equivalently in other systems $Y \leq z_{12} z_{23}$ or $z_{13} z_{23}$. Use of a single cut-off in the place of $\text{Min}(Y, z_{12} z_{13}, z_{12} z_{23}, z_{13} z_{23})$ leads to considerable simplifications in the calculations. Clearly, in regions where one of the products $z_{ij} z_{ik}$ becomes very small, i.e. where the 3-jet is no longer well defined, the approximation becomes problematic. The thrust (T) dependence of the size of higher order Y corrections (mentioned at the end of section 3.2) or the quality of the $O(Y^0)$ approximation can be traced back to the inequality (4.2.13). For

²See /52/ for another choice.

[†]another choice is

normalization factor N_v for the v -integration:

$$(4.2.19) \quad \int_0^1 dv \, v^{-\epsilon} (1-v)^{-\epsilon} = N_v \equiv \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}$$

In addition we want to extract two factors out of (4.2.17): first, a constant C arising in a natural way when calculating the virtual corrections (C is already defined in (3.2.18))

$$(4.2.20) \quad C \equiv \frac{1}{8\pi^2} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\frac{4\pi}{2} \right]^\epsilon \left(1 - \frac{\pi^2}{3} \epsilon^2 \right)$$

and, second, the phase space for three massless particles as is necessary when one adds the tree graph contributions to the virtual corrections:

$$(4.2.21) \quad dPS^{(3)} = \frac{q^2}{2\pi^3} dz_{13} dz_{23} [z_{12} z_{13} z_{23}]^{-\epsilon} \left[\frac{4\pi}{q^2} \right]^{2\epsilon} \frac{1}{\Gamma(2-2\epsilon)}$$

At the $O(y^0)$ level we can now write

$$(4.2.22) \quad dPS^{(4)} = dPS^{(3)} \int dR$$

where

$$(4.2.23) \quad \int dR = K_3 \int dy_{13} y_{13}^{-\epsilon} \left\{ d\hat{\Omega}_{13} \right\}$$

Here $d\hat{\Omega}_{13}$ is the spherical element in $(n-1)$ dimensions normalized

to unity $\left(\int_0^1 d\hat{\Omega}_{13} = 1 \right)$

$$(4.2.24) \quad d\hat{\Omega}_{13} = \frac{1}{N_v} \int dv \, v^{-\epsilon} (1-v)^{-\epsilon} \frac{1}{N_{\theta_1}} \int_0^\pi d\theta_1 \sin^{-2\epsilon} \theta_1$$

and

$$(4.2.25) \quad K_3 = \left[1 + \frac{\pi^2}{3} \epsilon^2 \right] \frac{C q^2 S}{2(1-2\epsilon)}$$

The curly brackets in $\{d\hat{\Omega}_{13}\}$ mean that proper care has to be taken to avoid double counting for overlapping singularities, see next section.

$T \rightarrow 1$ the R.H.S. of (4.2.13) goes to zero. For a fixed mass cut-off y the inequality $y \leq z_{12} z_{13}$ ($z_{12} z_{23}$ or $z_{13} z_{23}$) is no longer fulfilled as $T \rightarrow 1$ and one thus incurs an error. This error is of $O(y^0)$ but seems to be rather large for high thrust values T ($T > 0.95$; $0.01 \leq y \leq 0.04$).

The scaled dot products y_{ij} following from (4.2.11) can be

$$(4.2.14) \quad y_{24} = z_{23} + y_{13}$$

$$y_{12} = (z_{12} - y_{13})v, \quad y_{23} = (z_{12} - y_{13})(1-v)$$

$$y_{14} = (z_{13} - y_{13})(v + \delta), \quad y_{34} = (z_{13} - y_{13})(1-v - \delta)$$

where

$$(4.2.15) \quad v \equiv \frac{1}{2}(1 - \cos\theta)$$

$$(4.2.16) \quad \delta \equiv (1 - 2v)\lambda - 2\cos\theta_1 \sqrt{v(1-v)\lambda(1-\lambda)}$$

and λ as given in (4.2.12). Note that $\delta \rightarrow 0$ for $y_{13} \rightarrow 0$.

The 4-particle phase space in this 1-3 frame is given (after

integration of the angles $\theta_2, \dots, \theta_{n-3}$) by ($n \equiv 4 - 2\epsilon$)

$$(4.2.17) \quad dPS^{(4)} = \frac{q^4}{2^{11} \pi^5} \frac{S}{\Gamma(1-\epsilon) \Gamma(2-2\epsilon)} \left[\frac{4\pi}{q^2} \right]^{3\epsilon} \int dy_{123} dy_{134} dy_{13} (y_{123} y_{134} - y_{13})^{-\epsilon} (y_{13} + 1 - y_{123} - y_{134})^{-\epsilon} y_{13}^{-\epsilon} \int dv (v(1-v))^{-\epsilon} \frac{1}{N_{\theta_1}} \int_0^\pi d\theta_1 \sin^{-2\epsilon} \theta_1$$

S is the statistical factor and N_{θ_1} is a normalization factor

$$(4.2.18) \quad \int_0^\pi d\theta_1 \sin^{-2\epsilon} \theta_1 = N_{\theta_1} \equiv 2^\epsilon \pi \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)}$$

The statistical factor S and the integration limits appropriate for the respective singular regions will be discussed below.

For later use we find it convenient to define a similar

Here and in the following we suppress overall factors coming from the phase space. We note that the integrands do not depend on the angle θ_1 .

It turns out to be advantageous to add the integrands of the various class A-C contributions that have the same colour factors. Therefore explicit integration will be deferred after discussing the contributions of the other classes.

Class B

Class B possesses the same singularities as class A, namely Y_{13}, Y_{23}, Y_{14} and Y_{24} . However, we find ourselves confronted with "overlapping" divergences⁴.

Consider first the limit $Y_{13} \rightarrow 0$ at first. We obtain

$$(4.3.7) \quad S_{13}^{(i)B} \approx \frac{1}{Y_{13}} \left\{ 2 \left[\frac{1}{1-v} - 1 \right] S_{13}^{(i)}(z_{13}, z_{23}) - R_i(v, z_{jk}) \right\}$$

Again, the limit $Y_{23} \rightarrow 0$ can be obtained from (4.3.7) by $Y_{13} \rightarrow Y_{23}$ and $z_{13} \leftrightarrow z_{23}$ in R_i .

The denominator $(1-v)$ in (4.3.7) originates from a Y_{23}^- factor via $Y_{23} = (z_{12} - Y_{13})(1-v)$. We thus have to integrate over a phase space region where Y_{13} and/or Y_{23} can become small. Integration of both $S_{13}^{(i)B}$ and $S_{23}^{(i)B}$ over the whole range of v (in the 2-3 frame, v is defined by $Y_{13} = (z_{12} - Y_{13})(1-v)$) would lead to a double counting of the region $0 \leq Y_{13} \leq Y, 0 \leq Y_{23} \leq Y$ (compare the discussion of the 3-2-jet reduction in §3.2).

⁴Note that the integrand contains terms proportional to

$1/(Y_{13}Y_{23})$, while the integration region includes the limit $Y_{13} \rightarrow 0$ and $Y_{23} \rightarrow 0$ simultaneously.

Therefore in one of the v -integration (e.g. in the one for $S_{13}^{(i)B}$) the upper limit of v has to be changed from 1 to $(1 - Y/Z_{12})$.

(The range $0 \leq v \leq 1 - Y/Z_{12}$ corresponds to $Y \leq Y_{23} \leq Z_{12}$, and the upper bound Z_{12} follows immediately from the definition $Y_{23} = Z_{12} - Y_{12}$ remembering $Y_{12} \geq 0$).

The singular regions $Y_{14} \rightarrow 0$ and $Y_{24} \rightarrow 0$ have to be treated in the same way and yield simply a factor 2 which again gets canceled against the statistical factor S .

In total, the contribution of class B is

$$(4.3.8) \quad \text{class B} \rightarrow \int dY_{13} \int dY_{23} \int d\theta_1 \left[\begin{array}{ccc} Y & 1-Y/Z_{12} & \pi \\ 0 & 0 & 0 \end{array} \right] + \int dY_{23} \int d\theta_1 \left[\begin{array}{ccc} Y & 1 & \pi \\ 0 & 0 & 0 \end{array} \right] S_{13}^{(i)B} S_{23}^{(i)B}$$

Again we note that the integrands do not depend on θ_1 .

Class C

In the treatment of class C_I we have to be careful since here besides $Y_{13}, Y_{23}, Y_{14}, Y_{24}$ also Y_{34} appears in the denominator. Multiple overlapping divergences will then appear.

Let us first consider the limit $Y_{13} \rightarrow 0$. It contains also a Y_{34} singularity which appears as a term $1/(1-v)$ due to $Y_{34} \approx$

$$Z_{13}(1-v):$$

$$(4.3.9) \quad S_{13}^{(i)C} \approx \frac{1}{Y_{13}} \left\{ \left[\frac{1}{1-v} - 1 \right] S_{13}^{(i)}(z_{13}, z_{23}) - \frac{1}{Z} R_i(v, z_{jk}) \right\}$$

The same happens for $Y_{23} \rightarrow 0, Y_{14} \rightarrow 0$ and $Y_{24} \rightarrow 0$.

In the limit $Y_{34} \rightarrow 0$ each of the variables $Y_{13}, Y_{23}, Y_{14}, Y_{24}$ can lead to a singularity appearing as $1/v$ or $1/(1-v)$:

$$(4.3.10) \quad S_{34}^{(i)C} \approx \frac{4}{Y_{34}} \left\{ \frac{1}{Z} \left[\frac{1}{(1-v)v} - 1 \right] S_{13}^{(i)}(z_{13}, z_{23}) \right\}$$

Let us first consider the integration in the region $Y_{13} \rightarrow 0$ and/or $Y_{34} \rightarrow 0$. If we integrate in $S_{13}^{(i)C}$ the variable v , i.e. Y_{34} , from 0 to 1, then we have to restrict the v - (i.e. the Y_{13} -)

integration in $S_{34}^{(i)C_I}$ from 0 to $(1-y/z_{13})$, compare the discussion above. Similarly, in $S_{23}^{(i)C_I}$, v (or Y_{34}) ranges from 0 to 1 whereas in $S_{34}^{(i)C_I}$, v now is bounded by 0 and $(1-y/z_{23})$. After integration the terms coming from the phase space regions (Y_{14} and/or $Y_{34} \rightarrow 0$) and (Y_{24} and/or $Y_{34} \rightarrow 0$) yield the same contributions as the two cases discussed above. The factor 2 they induce is again canceled by S .

However, carrying out the integration this way would count the phase space region ($Y_{34} \rightarrow 0$) 4-times. Therefore we must divide the integrand (4.3.10) by a factor 4. Collecting these results we get the following integrations for class C_I (after the trivial θ_1 -integration; there is no θ_1 -dependence):

$$(4.3.11) \quad \left\{ \int_0^Y dy_{13} \int_0^v dv S_{13}^{(i)C_I} + \int_0^Y dy_{34} \int_0^{1-y/z_{13}} dv \frac{1}{4} S_{34}^{(i)C_I} \right\} + \left\{ 1 \leftrightarrow 2 \right\}$$

Class C_G

Class C_G contains the gluon loop graphs. Since we sum over the two physical polarizations of the produced gluons via

$$(4.3.12) \quad \sum_{\text{pol}}^* \epsilon_\alpha \epsilon_\beta = -g_{\alpha\beta}$$

we have to include "ghost loop diagrams" in order to cancel the propagation of unphysical degrees of freedom of the gluon⁵.

Class C_G becomes singular only in the phase space region $Y_{34} \rightarrow 0$. Thus the integration would simply be⁶

$$(4.3.13) \quad \text{class } C_G \rightarrow \frac{1}{2} \int_0^Y dy_{34} \int_0^1 dv S_{34}^{(i)C_G}$$

⁵The poor man's ghost prescription /36/ cannot be used since this prescription is no longer valid in n dimensions.

⁶The factor 1/2 is the statistical factor S .

However, some care is necessary in calculating the integrand. Since $S_{34}^{(i)C_G}$ contains Y_{34} double poles, the limiting process $Y_{34} \rightarrow 0$ is allowed only after cancellation of one of the poles against a Y_{34} factor in the numerator. In the p.c. case, all terms containing a Y_{34} double pole are proportional to the factor

$$(4.3.14) \quad F_{34} \equiv (Y_{13}Y_{24} - Y_{14}Y_{23})^2$$

Using the explicit representation of the momenta appropriate for the 3-4 frame (see C-1) and neglecting higher powers of Y_{34} , we find that F_{34} is proportional to Y_{34}

$$(4.3.15) \quad F_{34} \simeq Y_{34} Z_{12} Z_{13} Z_{23} 4v(1-v)\cos^2\theta_1$$

containing an additional explicit θ_1 -dependence.

Keeping in mind this subtlety we get the contribution from

class C_G

$$(4.3.16) \quad \text{class } C_G \rightarrow \frac{1}{2} \int_0^Y dy_{34} \int_0^v dv \frac{1}{N_{\theta_1}} \int_0^\pi d\theta_1 \sin^{-2\epsilon}\theta_1 \left\{ 2[v(1-v) - 1] S_{\text{Born}}^{(i)} + 4v(1-v) \frac{Z_{12}}{Z_{13}Z_{23}} [(1-\epsilon)2\cos^2\theta_1 - 1] \right\}$$

Here we have written the integrand already in a form showing the vanishing of the last term after performing the θ_1 -integration.

We observe that there is a factorization into the corresponding Born terms and an Altarelli-Parisi type kernel P_{qg} which is the same for all projections of the hadron tensor:

$$(4.4.4) \quad P_{qg}(v) \equiv T_R \frac{v^2 + (1-v)^2 - \epsilon}{1 - \epsilon}$$

4.4 Integration of $e^+e^- \rightarrow q\bar{q}q\bar{q}$

The contributions to the process

$$(4.4.1) \quad \gamma^*(q) \rightarrow q(p_1) + \bar{q}(p_2) + q(p_3) + \bar{q}(p_4)$$

fall into three classes (class D, E and F in ERT's notation)

distinguished by their group weight factors. The corresponding

Feynman diagrams are shown in Figs. 12, 13 and table 6.

Diagrams of class F give no contribution as a consequence of Furry's theorem. We also do not have to consider contributions to $\gamma^* \rightarrow q_A \bar{q}_A q_B \bar{q}_B$ from interference terms for identical flavours $q_A = q_B$ (class E) since these terms are finite in the limit of vanishing invariant pair masses Y_{ij} .

The squared matrix elements of class D with colour factor $C_F T_R$ ($T_R = N_f/2$ for N_f flavours) can be divided into four groups transformed into one another by permutation of momentum labels.

Since each group is singular only for a specific Y_{ij} going to zero, they all yield the same contribution after integration. It is thus sufficient to consider one group, e.g. the one with the Y_{34} -singularity, and to integrate it over

$$(4.4.2) \quad \int_0^Y \frac{1}{Y_{34}} dv \int_0^\pi d\theta_1 S_{34}^{(i)D}$$

The factor 4 coming from the other groups just cancels the statistical factor $S = 1/(2!2!)$ of process (4.4.1). Keeping in mind that the squared matrix elements possess Y_{34} double poles, we obtain:

$$(4.4.3) \quad S_{34}^{(i)D} = \frac{1}{Y_{34}} \frac{v^2 + (1-v)^2 - \epsilon}{1 - \epsilon} S_{\text{Born}}^{(i)}$$

after performing the θ_1 -integration in the limit $Y_{34} \rightarrow 0$.

4.5 Integrated p.c. 4-parton hadron tensor

In this subsection we present the final results for the real (tree) p.c. hadron tensor $H_{\mu\nu}^{(3)}(q_1, y)_{\text{Real}}$ after integration of the remaining variables y_{ij} and v in the 4-parton hadron tensor integrands. To this end we combine the individual (θ_1 -integrated) contributions according to their colour factors.

C_F -Terms

We have to add up the contributions of class A and B. We first observe that the upper limit of the v -integration of the first term in (4.3.5) can be changed from 1 to $(1-y/z_{12})$ without affecting the integrated result to $O(y^0)$ since the integrand does not contain any term which diverges as $v \rightarrow 1$. Then the integrations for class A and B coincide and after addition of the integrands only Born term like terms survive:

$$(4.5.1) \quad \int_0^y dy_{13} \int_0^{1-y/z_{12}} dv \frac{1}{y_{13}} \left\{ (1-\epsilon)(1-v) + \frac{2v}{1-v} \right\} S_{\text{Born}}^{(i)}(z_{13}, z_{23}) \\ + \int_0^y dy_{23} \int_0^{1-y/z_{12}} dv \frac{1}{y_{23}} \left\{ (1-\epsilon)(1-v) + \frac{2v}{1-v} \right\} S_{\text{Born}}^{(i)}(z_{23}, z_{13})$$

Using the symmetry of the Born terms under $1 \leftrightarrow 2$ and inserting all factors the final integral can be written as

$$(4.5.2) \quad S^{(i)}C_F = g^4 N_C^2 C_F^2 C_8 S_{\text{Born}}^{(i)} I_C$$

with

$$(4.5.3) \quad I_C = (1 + \frac{\pi^2}{3} - \epsilon) \frac{r(1-2\epsilon)}{r^2(1-\epsilon)} \int_0^y dy_{13} y_{13}^{-\epsilon} \\ \left\{ 2 \int_0^1 dv \int_0^{1-y/z_{12}} dv v^{-\epsilon}(1-v)^{-\epsilon} \frac{1}{y_{13}} \left\{ (1-\epsilon)(1-v) + \frac{2v}{1-v} \right\} \right\}$$

We find that the C_F term in the $O(\alpha_s^2)$ matrix element squared factorizes the Born term, i.e. has the kinematical structure of the $O(\alpha_s)$ matrix element: it is proportional to the matrix element squared in which particles one and three are considered as a single composite times an Altarelli-Parisi function

$$(4.5.4) \quad P_{qq}(v) = C_F \left\{ \frac{2}{1-v} - 1 - v - \epsilon(1-v) \right\}$$

N_C -Terms

To get the integrand for the colour factor $C_F N_C$ we have to add up the contributions of class C and $(-1/2)$ times the contributions of class B. In doing so we have to pay attention to the upper bounds of the v -integrations. Clearly, for terms which are finite for $v \rightarrow 1$, the values 1 and $(1-y/z_{12})$ give the same integral to $O(y^0)$. Keeping in mind this fact we find again that all non-Born term like terms vanish. Using the $1 \leftrightarrow 2$ symmetry of the Born terms the integrand proportional to the colour factor N_C is

$$(4.5.5) \quad S^{(i)}N_C = g^4 N_C^2 C_F^2 C_8 S_{\text{Born}}^{(i)} I_N$$

with

$$(4.5.6) \quad I_N = I_I + I_G$$

$$(4.5.7) \quad I_I = (1 + \frac{\pi^2}{3} - \epsilon) \frac{r(1-2\epsilon)}{r^2(1-\epsilon)} \left\{ \int_0^y dy_{13} y_{13}^{-\epsilon} \int_0^{1-y/z_{12}} dv v^{-\epsilon}(1-v)^{-\epsilon} \frac{1}{y_{13}} \frac{v}{1-v} \right. \\ \left. + \int_0^y dy_{34} y_{34}^{-\epsilon} \left[\int_0^{1-y/z_{13}} dv + \int_0^{1-y/z_{23}} dv \right] v^{-\epsilon}(1-v)^{-\epsilon} \frac{1}{y_{34}} \frac{1}{2} \left[\frac{1}{(1-v)} - 1 \right] \right\}$$

$$(4.5.8) \quad I_G = (1 + \frac{\pi^2}{3} - \epsilon) \frac{r(1-2\epsilon)}{r^2(1-\epsilon)} \\ \left\{ \int_0^y dy_{34} y_{34}^{-\epsilon} \int_0^{1-y/z_{34}} dv v^{-\epsilon}(1-v)^{-\epsilon} \frac{1}{y_{34}} \left[v(1-v) - 1 \right] \right\}$$

Again, after θ_1 -integration the result has the kinematical

structure of the lowest order Born contribution with partons 1 and 3 (3 and 4) interpreted as a single particle. We note that the last term (4.5.8) is finite for $v \rightarrow 1$. So its v -integration limits may be changed to the v -integration limits as they appear in the second line of (4.5.7). We then recognize the Altarelli-Parisi kernel P_{GG} in the integrand of the 3-4 integration

$$(4.5.9) \quad P_{GG}(v) = N_C \left\{ \frac{2}{v} + \frac{2}{1-v} - 4 + 2v(1-v) \right\}$$

The 1-3 singular contributions in (4.5.7), also proportional to N_C , are terms that are singular in the 3-jet region but make no contribution in the leading log approximation¹. They are therefore not related to Altarelli-Parisi kernels. The existence of these terms indicates that the leading log approximation may not be used to extract information about subleading logs.

T_R-Terms

The integrand appropriate to the colour factor $T_R \equiv N_f/2$ was already given in (4.4.3):

$$(4.5.10) \quad S^{(i)T_R} = g^4 N_C C_F T_R C \text{ 8 S Born } I_D$$

where

$$(4.5.11) \quad I_D = (1 + \frac{n^2}{3} \epsilon^2) \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \int_0^Y \int_0^Y \int_0^Y \int_0^Y \frac{1}{v^2} \frac{v^{2+(1-v)^2-\epsilon}}{1-\epsilon}$$

Integrated Results

The above integrations can be done without difficulties and

¹When naively adding Class B and C the 1-3 singular terms cancel.

after a bit of regrouping the final result is /73/

$$(4.5.12) \quad S^{(i)}(\text{real}) \equiv S^{(i)C_F} + S^{(i)N_C} + S^{(i)T_R} \\ = g^4 N_C C_F \left[C_F H_C + N_C/2 H_N + (N_f/3 - 11N_C/6) H_f \right] \text{ 8 S Born }^{(i)}$$

where

$$(4.5.13) \quad H_C = C \left\{ \frac{2}{\epsilon^2} - \frac{1}{\epsilon} (2 \ln z_{12} - 3) + 7 - 2 \ln^2 y + 4 \ln y \ln z_{12} \right. \\ \left. - \ln^2 z_{12} - 3 \ln y + \frac{\pi^2}{3} \right\}$$

$$H_N = C \left\{ \frac{2}{\epsilon^2} - \frac{2}{\epsilon} \ln \frac{z_{13} z_{23}}{z_{12}} + \frac{4}{3} - \ln^2 z_{13} - \ln^2 z_{23} + \right. \\ \left. \ln^2 z_{12} - 2 \ln^2 y + 4 \ln y \ln \frac{z_{13} z_{23}}{z_{12}} + \frac{\pi^2}{3} \right\}$$

$$H_f = C \left\{ -\frac{1}{\epsilon} - \frac{5}{3} + \ln y \right\}$$

One notes from (4.5.12) that all projections of the integrated p.c. $O(\alpha_s^2)$ 4-parton hadron tensor factorize to $O(y^0)$ into the corresponding 3-parton Born (i.e. α_s^1) terms and an

universal IR factor (i.e. a factor which is the same for all projections of the hadron tensor). Since the hadron tensor is fully determined by the set of all its projections we can write (4.5.12) in a form showing this universal behaviour explicitly:

$$(4.5.14) \quad H_{\mu\nu}^{(3) \text{ p.c. } (q_1, q_2, q_3; Y)} \text{ Real} = \\ g^2 \left[C_F H_C + N_C/2 H_N + (N_f/3 - 11N_C/6) H_f \right] 4g^2 N_C C_F B_{\mu\nu}^{(3) \text{ p.c. } (q_1, q_2, q_3)}$$

The 3-parton Born hadron tensor $B_{\mu\nu}^{(3)}$ is given in (4.1.10). (Note that $g^2 C \rightarrow \alpha_s/(2\pi)$ for $\epsilon \rightarrow 0$).

5. THE P.V.-REAL (TREE GRAPH) CONTRIBUTIONS

5.1 General remarks

Calculations of p.v. quantities involving parity-odd fermion traces (i.e. traces with an odd number of γ_5 's) require a careful discussion of how to deal with γ_5 or the totally antisymmetric tensor $\epsilon(\alpha, \beta, \gamma, \delta)$ in n dimensions /83/. The n-dimensional parity-odd Dirac traces generate $O((n-4)^m; m>0)$ anomalous contributions, which in turn lead to finite anomalous terms when multiplied by ultraviolet (UV) or IR/M divergent integrals. Besides, if one does not use a consistent dimensional regularization scheme the result will be ambiguous. As discussed in detail in Appendix A and B a consistent scheme of dimensional regularization including γ_5 is provided by the BM γ_5 -scheme /25/ which will be used in the following.

Use of a consistent γ_5 -scheme leads to unique results, i.e. to results which are free of ambiguous terms. However, during the calculation there arise anomalous terms. In the context of $O(\alpha_s^2)$ calculations we encounter IR/M anomalies which violate

$$(5.1.1) \quad \begin{aligned} &\text{-chiral symmetry} && H_{\mu\nu}^{VA} \neq H_{\mu\nu}^{AV} \\ &\text{-gauge invariance} && q^\mu H_{\mu\nu}^{P.V.} \neq 0 \\ &\text{-charge conjugation (C) symmetry} && H_{\mu\nu}^{P.V.}(q, \bar{q}) \neq -H_{\mu\nu}^{P.V.}(\bar{q}, q) \end{aligned}$$

The treatment of UV anomalies is well known /23,5,79/. They are spurious in our case and must be cancelled by taking the appropriate renormalization scheme /5,22,24,104,13/¹. The

¹See also [6].

occurrence of IR/M anomalies in the odd parity fermion loops in the dimensional regularization scheme has first been discussed in /64/. Up to now calculations were carried out either for total cross sections (no V-A terms) or for lower order calculations where these problems did not arise. Other authors tried to parametrize the ambiguities /30,31/. Of course when regularization schemes with regulator masses are used these problems are avoided since one no longer needs a continuation in space-time from $n=4$ in order to regularize the UV and IR/M infinities.

In /66/ we did a first systematic analysis of the dimensional IR/M regularization for p.v. cross sections. Essentially, there are three² different versions of dimensional regularization (see Appendix A). We found that IR/M anomalies can not be avoided, they arise in each scheme. Therefore additional terms are required when expanding integrals into a complete set of covariants.

Presently there is only one systematic and consistent formulation of dimensional regularization, the one of BM /25/. This version regularizes UV and IR/M divergent and anomalous quantities in a unique way.

It has been demonstrated in /25,26,104/ that the BM scheme provides a consistent treatment of the singularities in the UV

²The question is open whether the γ_5 -scheme of /103/ provides a consistent formulation of the γ_5 -problem. Work is in progress whether the non-uniqueness of their parameter b leads to a unique result /1/.

realm (correct anomalies, all order renormalizability).

That the BM scheme also provides a consistent and practical calculational scheme for the treatment of IR/M singularities will be demonstrated in the following by calculating through sufficiently complex examples.

First note that in parity- odd traces the γ_5 - substitution (A31) yielding long traces need not be done explicitly if one works with suitable parity- odd projection operators³.

Second, there is no need to use the "ugly" anticommutation relation (A32) since all trace manipulations can be performed (at least if there is only one γ_5 present) without commuting γ_5 because of the cyclic property of the trace.

Finally, the action of the parity- odd projection operators produce "4-dimensional" scalars via (A28). These have to be treated separately from the "n-dimensional" scalars resulting from the trace manipulations. As will be clear in the next sections, there will be only one relevant "4-dimensional" scalar for every IR/M region which the computer can easily handle.

In order that the BM scheme can be implemented successfully on a computer, it is desirable that non-"n-dimensional" quantities come into play only in a definite way (e.g. via products of ϵ -tensors). Furthermore, since γ_5 itself is needed and since a computer program like REDUCE handles only "4-dimensional" γ_5 's, we

³Provided there is at least one free Lorentz index. But the number of independent traces of the form $\text{Tr} \gamma_5 p_1 \dots p_{2k}$ is normally small, where p_i are all linearly independent for $k \geq 3$.

must take care that it does not use e.g. the wrong Schouten identity.

Anticipating the results of calculating the p.v. 3-jet hadron tensor we will see that the IR/M anomalies appearing in the $O(\alpha_s^2)$ calculation do cancel among real and virtual contributions.

However, a general proof for the assumption that IR/M anomalies are always spurious in the above sense is still missing.

Nevertheless, the expertise gained from this first explicit calculation of a higher order QCD correction to a p.v. cross section will be quite valuable for many higher order calculations that have to be done for the interpretation of the p.v. experiments at high- energy machines to be completed in the following years.

5.2 The $O(\alpha_s)$ p.v. 2-jet cross section $e^+e^- \xrightarrow{\gamma, Z} q\bar{q}$

In this section we present the results of calculating the $O(\alpha_s)$ corrections to the p.v. 2-jet cross section. Many of the additional difficulties arising in the calculation of p.v. cross sections within dimensional regularization already appear at this $O(\alpha_s)$ level. We therefore present more details of the $O(\alpha_s)$ calculation in order to clarify important technical issues relevant for the technically more involved $O(\alpha_s^2)$ calculation.

First note that in $n \neq 4$ dimensions chiral invariance is violated so that in general $H_{\mu\nu}^{VA} \neq H_{\mu\nu}^{AV}$ (compare (2.3.2)). This implies that $H_{\mu\nu}^{P.V.}$ has to be calculated from

$$(5.2.1) \quad H_{\mu\nu}^{P.V.} \equiv H_{\mu\nu}^4 \equiv \frac{1}{2} (H_{\mu\nu}^{VA} + H_{\mu\nu}^{AV}) \neq H_{\mu\nu}^{VA}$$

Our task is therefore to evaluate $H_{\mu\nu}^{P.V.}$ [2-jet; α_s], given as the sum of real and virtual contributions¹

$$(5.2.2) \quad \begin{aligned} H_{\mu\nu}^{P.V.} [2\text{-jet}; \alpha_s] &\equiv H_{\mu\nu}^{(2)P.V.}(r_1, r_2) \\ &= H_{\mu\nu}^{(2)P.V.}(r_1, r_2)_{\text{virt}} + H_{\mu\nu}^{(2)P.V.}(r_1, r_2)_{\text{Real}} \end{aligned}$$

where

$$(5.2.3) \quad H_{\mu\nu}^{(2)P.V.}(r_1, r_2)_{\text{virt}} \equiv H_{\mu\nu}^{P.V.} [2\text{-partons}; \alpha_s]$$

and to order y^0 we obtain

$$(5.2.4) \quad H_{\mu\nu}^{(2)P.V.}(r_1, r_2)_{\text{Real}} \equiv \frac{q^2}{16\pi^2} \left[\frac{4\pi\mu^2}{q^2} \right]^{\epsilon} \frac{1}{F(1-\epsilon)} \int_0^y dz_{23} z_{23}^{-\epsilon} \left\{ \int dv \right\} H_{\mu\nu}^{P.V.} [3\text{-partons}; \alpha_s]$$

The integration symbol $\left\{ \int dv \right\}$ stands for (see §3.2)

$$(5.2.5) \quad \left\{ \int dv \right\} = \left[\int_0^1 dv - \int_0^1 dv \right] y^{(1-Z_{23})} v^{-\epsilon} (1-v)^{-\epsilon}$$

We note that due to definition (5.2.1) the real (tree graph) contribution to the p.v. hadron tensor is $\mu \leftrightarrow \nu$ antisymmetric, i.e.

$$(5.2.6) \quad H_{\mu\nu}^{P.V.}(r_1, r_2)_{\text{Real}} = -H_{\nu\mu}^{P.V.}(r_1, r_2)_{\text{Real}}$$

The $\mu \leftrightarrow \nu$ antisymmetry is true regardless of the space time dimension n since it does not depend on the commutation property of γ_5 .

Real $O(\alpha_s)$ contributions

As in the p.c. case we denote the momenta occurring in the 3-parton $\rightarrow$ 2-jet reduction by $(q_1, q_2, q_3) \rightarrow (r_1, r_2)$. Since the r_i are momenta of the 2-jet hadron tensor and therefore external momenta we must choose them to be 4-dimensional ($r_i \equiv \tilde{r}_i$), whereas the q_j in general have to be n -dimensional ($q_j \equiv \tilde{q}_j + \tilde{q}_j$) since they are integrated over in n dimensions. To be definite we consider the "2-3 system"

$$(5.2.7) \quad q \equiv q_1 + q_2 + q_3 \equiv k_1 + k_2$$

where

$$(5.2.8) \quad q_1 = \frac{q^2 - t_{23}}{2\sqrt{t_{23}}} (1, 0, \dots, 0, 1)$$

$$\begin{aligned} q_2 &= \frac{1}{2} \sqrt{t_{23}} (1, \dots, -\sin\theta_1, -\cos\theta) \\ q_3 &= \frac{1}{2} \sqrt{t_{23}} (1, \dots, +\sin\theta_1, +\cos\theta) \end{aligned}$$

and

$$(5.2.9) \quad k_1 \equiv q_1, \quad k_2 \equiv q_2 + q_3$$

We use the usual notation $z_{ij} \equiv t_{ij}/q^2$, $t_{ij} \equiv (q_j + q_j)^2 \equiv 2q_i q_j$, $v \equiv (1 - \cos\theta)/2$ so that $z_{13} = (1 - z_{23})v$.

¹ Compare section 3.2.

² The notation $p, \tilde{p}, \tilde{p}$ etc. is explained in App. A, see (A26, 27).

Note that we have to be careful since the 2-jet momenta r_i have to be on-mass-shell, i.e. $r_i^2 \stackrel{!}{=} 0$, while the k_i are in general off-shell, i.e. they are not yet the right 2-jet momenta r_i , they still depend on z_{23} .

To illustrate the connection between r_i and k_j , we first note that the RHS of (5.2.4) depends solely on k_1 and k_2 after performing the angular integration $\int d\mathbf{v}$. We write it in an obvious notation

$$(5.2.10) \quad \begin{aligned} & H_{\mu\nu}^{(2)p.v.}(r_1, r_2)_{\text{Real}} \\ &= C_2 \int_0^Y dz_{23} z_{23}^{-\epsilon} \left\{ \int d\mathbf{v} \right\} H_{\mu\nu}^{(3)p.v.}(q_1, q_2, q_3) \\ &\stackrel{!}{=} C_2 \int_0^Y dz_{23} z_{23}^{-\epsilon} \sum_j H_j^R(z_{23}) H_{\mu\nu,j}^{p.v.}(k_1, k_2) \end{aligned}$$

where $C_2 = \frac{q^2}{16\pi^2} \left[\frac{4\pi\mu^2}{q^2} \right]^\epsilon r^{-1}(1-\epsilon)$.

Apart from the z_{23} -dependence in the structure functions H_j^R there is also the z_{23} -dependence in the tensors $H_{\mu\nu,j}^{p.v.}(k_1, k_2)$ themselves, since $k_2^2 = t_{23}$. However, this need not concern us at the $O(y^0)$ level because of the following reason: It is convenient to do the z_{23} -integration in the e^+e^- cms. Defining the z -axis by the off-mass-shell momentum k_2 we have

$$(5.2.11) \quad \begin{aligned} k_{2\text{cms}} &= (k_{20}, 0, 0, \sqrt{k_{20}^2 - t_{23}}) \\ &= k_{20}(1, 0, 0, 1 - 2\frac{t_{23}}{q^2} + \dots), \quad k_{20} = \frac{\sqrt{q^2}}{2} \end{aligned}$$

and similarly

$$(5.2.12) \quad k_{1\text{cms}} = k_{10}(1, 0, 0, -1) = \frac{\sqrt{q^2}}{2}(1, 0, 0, 1) + O(z_{23}^2)$$

Thus the t_{23} -dependence ($t_{23} \equiv z_{23}q^2$) of the momenta k_i and thereby the t_{23} -dependence of the tensor $H_{\mu\nu,j}^{p.v.}(k_1, k_2)$ can be

neglected at the $O(y^0)$ level³ and we can identify $r_i \equiv k_i$.

The p.v. 3-parton hadron tensor in $O(\alpha_s)$ $H_{\mu\nu}^{(3)p.v.}(q_i)$ can be evaluated explicitly keeping in mind that

$$a) \left\{ \gamma_i, \gamma_5 \right\} = 2\gamma_i \gamma_5$$

b) the momenta q_i are n -dimensional. Thus the right Schouten identity (A30) has to be used (see the term $R_{\mu\nu}$ below).

$$c) H_{\mu\nu}^{VA} \neq H_{\mu\nu}^{AV}$$

Then we obtain

$$(5.2.13) \quad H_{\mu\nu}^{(3)p.v.}(q_1, q_2, q_3) = 4g^2 C_F N_C \left\{ B_{\mu\nu}^{(3)p.v.}(q_1, q_2, q_3) + \epsilon R_{\mu\nu} \right\}$$

where

$$(5.2.14) \quad B_{\mu\nu}^{(3)p.v.}(q_1, q_2, q_3) = \left\{ \sum_{k=6}^7 H_k(\text{Born}) H_{\mu\nu}^i + H \cdot \frac{i}{q^2} \epsilon(\mu\nu q_1 q_2) \right\}$$

$$(5.2.15) \quad \begin{aligned} R_{\mu\nu} &= \frac{-2i}{t_{13} t_{23}} \left\{ 2q_{3\mu} \epsilon(\nu q_1 q_2 q_3) - 2q_{3\nu} \epsilon(\mu q_1 q_2 q_3) \right. \\ &\quad \left. + (t_{13} + t_{23}) \epsilon(\mu\nu q_1 q_2) + t_{23} \epsilon(\mu\nu q q_1) - t_{13} \epsilon(\mu\nu q q_2) \right\} \end{aligned}$$

The covariants $H_{\mu\nu}^6 \equiv i\epsilon(\mu\nu q q_1)/q^2$ and $H_{\mu\nu}^7 \equiv i\epsilon(\mu\nu q q_2)/q^2$ were already defined in (2.3.10) when we discussed the most general expansion of a 3-particle hadron tensor. The two structure functions H_8 and H_9 appearing in the general expansion (2.3.10) are not present since we are considering a real tree graph contribution. $H_6(\text{Born})$ and $H_7(\text{Born})$ are given by

³ It should be noted, though, that the IR/M singular pieces are picked up at the on-mass-shell limit $k_i^2 = 0$ regardless of the $O(y^0)$ approximation. This is of course quite necessary in order to cancel the IR/M singular pieces of the loop contributions which occur for on-mass-shell momenta.

$$\begin{aligned}
 (5.2.16) \quad H_6(\text{Born}) &= 2 \frac{1-z_{23}}{z_{13}z_{23}} - \frac{2\epsilon}{t_{23}} \\
 H_7(\text{Born}) &= -H_6(\text{Born}; 1 \leftrightarrow 2)
 \end{aligned}$$

H' , defined by

$$(5.2.17) \quad H' = -2\epsilon \frac{1-z_{12}}{z_{13}z_{23}}$$

shows that there are additional "anomalous" covariants compared to (2.3.10). The covariant associated with H' corresponds to an anomaly that violates gauge invariance ($q^\mu \epsilon(\mu\nu q_1 q_2) \neq 0$), while the quantity $R_{\mu\nu}$ originates from an anomaly that violates the chiral symmetry ($R_{\mu\nu}$ obtains different contributions from H^{VA} and H^{AV}).

Due to the fact that $\left\{ \gamma_\alpha, \gamma_5 \right\} \neq 0$ in n dimensions the axial vector current is anomalous in the sense that $C(J_\mu^A) C^{-1} \neq J_\mu^A$ implying $H^{\text{VA}} \neq H^{\text{AV}}$. However, in this $O(\alpha_s)$ calculation this charge conjugation anomaly vanishes when taking the sum $H^{\text{p.v.}} = \frac{1}{2}(H^{\text{VA}} + H^{\text{AV}})$.

We will see later that the two other anomalous terms

$(H' \epsilon(\mu\nu q_1 q_2)$ and $R_{\mu\nu})$ do vanish after integration. This is different from what we will find in the $O(\alpha_s^2)$ calculation (see §5.3), where the gauge dependent terms survive the integrations.

We will now show that the term $R_{\mu\nu}$ can be dropped. To this end we use the Schouten identity for the 4-dimensional part $\tilde{q}$ of the momenta $q \equiv \tilde{q} + \bar{q}$:

$$\begin{aligned}
 (5.2.18) \quad 0 &\equiv \frac{2i}{t_{13}t_{23}} \left\{ \tilde{q}_{3\mu} \epsilon(\nu q_1 q_2 q_3) - 2\tilde{q}_{3\nu} \epsilon(\mu q_1 q_2 q_3) \right. \\
 &\quad \left. + 2\tilde{q}_1 \cdot q_3 \epsilon(\mu\nu q_1 q_2) + 2\tilde{q}_2 \cdot q_3 \epsilon(\mu\nu q q_1) - 2\tilde{q}_1 \cdot q_3 \epsilon(\mu\nu q q_2) \right\}
 \end{aligned}$$

The crucial point is the property of the metric tensors

$$(5.2.19) \quad g_\mu^\rho \tilde{q}_{\rho\nu} = \tilde{g}_{\mu\nu}$$

i.e. the relation $\tilde{a} \cdot b = a \cdot \tilde{b} = \tilde{a} \cdot \tilde{b}$ for any vectors a and b . Using then the 4-dimensionality of the momenta k_i and q implying $q_1 = k_1 = \tilde{q}_1$ and $q_2 + q_3 = k_2 = \tilde{q}_2 + \tilde{q}_3$ we get

$$\begin{aligned}
 (5.2.20) \quad 2\tilde{q}_1 \cdot q_3 &= 2q_1 q_3 = t_{13}, \quad 2\tilde{q}_1 \cdot q_3 = t_{13} + t_{23} \\
 2\tilde{q}_2 \cdot q_3 &= 2(\tilde{k}_2 - \tilde{q}_3) \cdot q_3 = t_{23} - 2\tilde{q}_3^2
 \end{aligned}$$

The term $R_{\mu\nu}$ thus reduces to

$$(5.2.21)$$

$$R_{\mu\nu} = \frac{2i}{t_{13}t_{23}} \left\{ 2\tilde{q}_{3\mu} \epsilon(\nu q_1 q_2 q_3) - 2\tilde{q}_{3\nu} \epsilon(\mu q_1 q_2 q_3) + 2\tilde{q}_3^2 \epsilon(\mu\nu q q_1) \right\}$$

However, when H^R in (5.2.10) is calculated the momenta $q_{i\mu}$ will not contribute when contracted with an ϵ -tensor. The term $\tilde{q}_3^2$ is (see App. C) proportional to $t_{23}v(1-v)\sin^2\theta_1\sin^2\theta_2$. Taking into account that there is an additional angle dependence (θ_2) compared to usual n -dimensional scalars we find that $R_{\mu\nu}$ is proportional to $\epsilon(1-v)\epsilon(\mu\nu q q_1)/(1-z_{23})$ so that $\epsilon R_{\mu\nu}$ can be safely dropped.

In order to avoid tensor integrations we make use of the ansatz (5.2.10) for the angular integral in (5.2.5)

$$(5.2.22) \quad \left\{ d\nu \right\} H^{(3)\text{p.v.}}(q_1, q_2, q_3) = H^R H^{\text{p.v.}}(r_1, r_2)$$

We have used the fact that we may replace $k_i \rightarrow r_i$ to $O(y^0)$ and that there are only 2 independent momenta r_1 and $q = r_1 + r_2$ at hand. Therefore there exists only one covariant

$$(5.2.23) \quad H^{\text{p.v.}}(r_1, r_2) = i \epsilon(\mu \nu q r_1)$$

and therefore one invariant H^R . The covariant (5.2.23) is necessarily conserved. Thus we have

$$(5.2.24) \quad \left\{ d\nu \right\} q^\mu H^{(3)\text{p.v.}}(q_1, q_2, q_3) = 0$$

Although the integrand in (5.2.24) is nonzero, i.e.

$H^{(3)\text{p.v.}}(q_1, q_2, q_3)$ is not conserved, the integral vanishes. This means that the anomalous piece that violates the gauge invariance vanishes after integration.

Contracting both sides of (5.2.22) with $\epsilon(\mu \nu q r_1)$ we finally arrive at

$$(5.2.25) \quad H^R = 4g^2 C_F^W \left\{ \left\{ dv \right\} \frac{2}{q^2} \frac{1}{z_{23}} \left[(1-\epsilon)v + 2\frac{1-v}{v} \right] \right.$$

Comparison of (5.2.25) with the corresponding integrand in the p.c. calculation in (3.2.16) reveals that both integrands coincide. This allows us to write the result for both p.c. and p.v. calculations in a compact way. We have ($X = p.c., p.v.$)

$$(5.2.26) \quad H_{\mu\nu}^{(2)X}(r_1, r_2)_{\text{Real}} = B_{\mu\nu}^{(2)X}(r_1, r_2) g^2 N_C^C C_F^C \left\{ \frac{2}{\epsilon} + \frac{3}{\epsilon} + 7 - 2\ln^2 y - 3\ln y + \frac{\pi^2}{3} \right\}$$

where the p.c. $O(\alpha_s^0)$ (Born) hadron tensor is defined in (3.2.14)

and

$$(5.2.27) \quad B_{\mu\nu}^{(2)p.v.}(r_1, r_2) = 4i \epsilon(\mu \nu q r_1)$$

The constant C in (5.2.26) is the same as in (3.2.18).

Virtual $O(\alpha_s)$ contributions

The vector and axial vector current contributions to the respective loop- amplitudes are given by $C_V \bar{u}(r_1) \gamma_\mu v(r_2)$ and $C_A \bar{u}(r_1) \gamma_\mu \gamma_5 v(r_2)$. C_V has been calculated in §3.2. In order to obtain the axial vector contribution C_A we impose the chiral invariance relation $C_V = C_A$.

Adding up the real and virtual contributions, we obtain the $O(y; \alpha_s)$ corrections to the 2-jet hadron tensor. Again we use the compact notation incorporating both the p.c. and the p.v. cases:

$$(5.2.28) \quad H_{\mu\nu}^X[2\text{-jet}; \alpha_s] = H_{\mu\nu}^X[2\text{-partons}; \alpha_s^0] \frac{\alpha_s}{2\pi} N_C^C C_F^C \left\{ -1 - 2\ln^2 y - 3\ln y + \frac{\pi^2}{3} \right\} \quad X = p.c., p.v.$$

In order to compare our p.v. results with the result of /59/ where the integration was performed over the whole 3-parton phase space we integrate the p.v. $q\bar{q}g$ contribution over the genuine

3-jet region outside the cut ($z_{13}, z_{23} \geq y$). We obtain

$$(5.2.29) \quad \sigma_6^{p.v.}(\alpha_s) = \sigma_6^{p.v.}(\alpha_s^0) \frac{\alpha_s}{2\pi} C_F \left\{ 1 + 2\ln^2 y + 3\ln y - \frac{\pi^2}{3} \right\}$$

The connection of the cross section σ_6 to the structure function

H_6 and H_7 appearing in $H_{\mu\nu}^{p.v.}(3\text{-partons}; \alpha_s)$ has been given in (2.3.23). $\sigma_6^{p.v.}(\alpha_s^0)$ is the corresponding cross section for the $q\bar{q}$ final state. We see that the sum of 2- and 3-jet contribution to σ_6 vanishes. This agrees with the $m_q \rightarrow 0$ result in /59/ where σ_6 was calculated as a function of the quark mass.⁴

Based on the result that the anomalies arising in intermediate steps of the calculation are spurious so that they cancel among virtual and real contributions, we conclude that the relation (2.3.3) $H_{\mu\nu}^3 \equiv i(H_{\mu\nu}^{VA} - H_{\mu\nu}^{AV}) = 0$ will remain unchanged for the IR/M regularized cross section.

⁴ In the calculation /59/ a fully anticommuting γ_5 was used within dimensional regularization. The γ_5 -results were "fixed" by a comparison with dispersion methods.

5.3 Integration of the p.v. $O(\alpha_s^2)$ 4-parton cross sections

Our aim is to calculate the real (tree) contributions to the p.v. 3-jet hadron tensor up to order y^0 . This is achieved by integrating the p.v. 4-parton process over the phase space regions of unresolved parton pairs to that accuracy. The techniques of IR/M integration (like dividing the diagrams into classes, choosing those diagrams which actually have to be integrated, splitting of the phase space) have been presented in detail in §4 for the p.c. case. Here we are dealing with a more complex case. Firstly, the scalarizing of the tensorial integrands is not so straightforward as in the p.c. case because of the appearance of anomalous contributions in connection with γ_5 and dimensional regularization. Secondly, we have the added complexity of odd parity fermion loops. These added complexities bring in new technical features which have to be discussed carefully.

Some of the additional difficulties did arise already in connection with the p.v. $O(\alpha_s)$ calculation in §5.2. To $O(y^0)$ we obtain for the tree graph contributions

$$(5.3.1) \quad \int_0^y dY_{13} \int_{13}^{\bar{e}} \left\{ d\hat{n}_{13} \right\} H_{\mu\nu}^{(4)p.v.}(p_1, p_2, p_3, p_4) \\ = H_{\mu\nu}^{(3)p.v.}(q_1, q_2, q_3; Y) \text{ real} =$$

The normalized spherical element $d\hat{n}_{13}$ and the factor K_3 were given in (4.2.24, 25), where we also explained the meaning of the curly brackets $\left\{ \right\}$. The 4-parton momenta p_i have to be chosen n-dimensional when integrated over, whereas the coordinate system must be chosen such that the 3-parton momenta q_j have only 4-dimensional components. In the $O(y^0)$ approximation they can be

chosen as a certain combination of the 4-parton momenta (e.g. in the 1-3 system: $q_1 \equiv p_1 + p_3$, $q_2 \equiv p_2$, $q_3 \equiv p_4$). To $O(y^0)$ they can be treated as on-mass-shell ($q_j^2 = 0$), see the discussion in §5.2.

The straightforward calculation of $H_{\mu\nu}^{(4)p.v.}$ as it was done for $H_{\mu\nu}^{(3)p.v.}$ in (5.2.13) would be too unwieldy. The calculation becomes very cumbersome due to the subtle properties of γ_5 and the ϵ -tensor in n dimensions. Either one could attempt to use the "ugly" γ_5 commutation properties (A32) to reduce the traces or one could replace γ_5 by its definition (A31) yielding extreme long traces (the 4-parton tensor contains up to 12 γ -matrices plus the γ_5). Both methods are unnecessarily cumbersome.

Instead of integrating tensor quantities we first expand (5.3.1) into a complete set of suitable covariants paying attention to include all possible anomalous covariants. One has to take into account that $\left\{ \gamma_\alpha, \gamma_5 \right\} \neq 0$ and that the Schouten-identity is valid only in the form (A30). Therefore the RHS of (5.3.1) develops anomalous terms in $n \neq 4$ which show up as

(i) chiral anomalies: $H_{\mu\nu}^{VA} \neq H_{\mu\nu}^{AV}$ (as an example see the term $R_{\mu\nu}$ in the $O(\alpha_s)$ calculation)

(ii) gauge anomalies: $q_{\mu}^{\mu} H_{\mu\nu}^{p.v.} \neq 0 \neq H_{\mu\nu}^{p.v.} q^{\nu}$

(iii) charge conjugation anomalies resulting from $C J_{\mu}^A C^{-1} \neq J_{\mu}^A$.

These anomalies induce additional terms in the most general expansion of the LHS of (5.3.1):

(5.3.2)

$$H_{\mu\nu}^{(3)p.v.}(q_1, q_2, q_3; Y) = H_6 \epsilon(\mu\nu q q_1) + H_7 \epsilon(\mu\nu q q_2) + H' \epsilon(\mu\nu q_1 q_2)$$

Here we have used the fact that $H_{\mu\nu}^{p.v.}$ is antisymmetric in $\mu \leftrightarrow \nu$ (see 5.2.6). Moreover, since the tree diagrams are real, there is

no contribution proportional to the covariants associated with the (imaginary) structure functions H_8 and H_9 , compare (2.3.10). We have also used the 4- dimensionality of the momenta q_i which allows us to use the Schouten identity to eliminate further covariants. This explains why there is no term like $R_{\mu\nu}$ corresponding to the chiral anomaly in (5.3.2) as in (5.2.13).

Note that the gauge anomaly is associated with the appearance of the structure function H' . The anomalous charge conjugate pieces in $H^{(4)}_{\mu\nu}$ resulting from $C(\gamma_\mu \gamma_5) C^{-1} \neq (\gamma_\mu \gamma_5)^T$ could in principle lead to violation of the charge conjugation symmetry $H_7(1,2) = -H_6(2,1)$ and $H'(1,2) = H'(2,1)$. However, we will see that in fact these anomalies vanish after the angular integration.

There is another, technical, point which distinguishes the p.v. and the p.c. case: The contractions of the p.c. projections (e.g. $q_{i\mu} q_{i\nu}$) with the p.c. 4-parton hadron tensor can be done without any difficulty, whereas the p.v. projections always include an ϵ -tensor which cannot be contracted with the p.v. hadron tensor until the latter is reduced to ϵ -tensors, too. However, this is the situation we want to avoid, since it corresponds to evaluating the total p.v. 4-parton hadron tensor (in n dimensions). Fortunately, a clever choice of the p.v. projections will considerably simplify the calculations.

First of all we note that there exists an expansion analogous to (5.3.2) for the angle integrated 4-parton tensor:

$$(5.3.3) \quad \left\{ \left\{ d\hat{n} \right\} H_{\mu\nu}^{(4)p.v.}(p_i) = F_6 \epsilon(\mu\nu q_1) + F_7 \epsilon(\mu\nu q_2) + F' \epsilon(\mu\nu q_1 q_2) \right.$$

To $O(\gamma^0)$, H_6 , H_7 and H' are related to F_6 , F_7 and F' by

$$(5.3.4) \quad H_i = K_3 \int_0^y dy_{13} y_{13}^{-\epsilon} F_i \quad (i=6,7); \quad H' = K \int_0^y dy_{13} y_{13}^{-\epsilon} F'$$

Instead of using $\epsilon(\mu\nu q_1)$ etc. as p.v. projectors, we scalarize the integrand in (5.3.3) by using the projectors

$$(5.3.5) \quad \begin{aligned} q_{2\nu} \epsilon(\mu q_1 q_2 q_3) &\rightarrow F_6 \\ q_{1\nu} \epsilon(\mu q_1 q_2 q_3) &\rightarrow F_7 \\ q_\nu \epsilon(\mu q_1 q_2 q_3) &\rightarrow F' \end{aligned}$$

which at the same time project out the three invariants F_6 , F_7 and F' as indicated in (5.3.5). Note that the contractions of (5.3.5) with the LHS of (5.3.3) can be exchanged with the angle integration in (5.3.3) since the q_i do not depend on the angular integration variables.

We proceed now as follows:

- 1) We rearrange each parity odd trace such that it starts with a γ_5 : $\text{Tr } \gamma_5 \left\{ \alpha \dots \gamma_\beta \dots \gamma_\alpha \dots \gamma_\beta \dots \gamma_\mu \dots \gamma_\nu \right\}$
- 2) The curly bracket is contracted with $q_{2\nu}$, $q_{1\nu}$ or q_ν , the first part of the respective projector of (5.3.5) (i.e. with $p_{2\nu}$, $p_{1\nu} + p_{3\nu}$, or $p_{1\nu} + p_{2\nu} + p_{3\nu} + p_{4\nu}$ in the 1-3 system).
- 3) Then we perform (on the computer with e.g. REDUCE) all contractions as well as other simplifications like

$$\gamma p_i \cdot \gamma p_j = -\gamma p_j \cdot \gamma p_i + 2 p_i \cdot p_j, \quad p_i^2 = 0$$

- in n dimensions. This is possible for REDUCE since γ_5 is not yet present (otherwise REDUCE would refuse to work in n dimensions).
- 4) After performing step 2 there are only 5 independent vectors available (p_1, p_2, p_3, p_4 and the index μ). Since a trace of γ_5 with 5 γ -matrices is zero we are left with traces containing only γ_5 and 4 γ -matrices. But these traces are the same in n and in 4 dimensions

$$\text{Tr } \gamma_5 \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta = 4i \epsilon(\alpha\beta\gamma\delta)$$

Switching to 4 dimensions is now allowed and we can multiply with γ_5 and take the trace on the computer.

5) What remains to be done is to contract with the rest of the projectors (5.3.5), i.e., with $\epsilon(\nu q_1 q_2 q_3)$. This must be done in 4 dimensions (see (A28)) and brings about 4-dimensional scalars via the product of ϵ -tensors (see (A28)). These 4-dimensional scalars have to be treated separately¹ from the n-dimensional scalars resulting from the trace manipulations. However, this does not pose a big problem, since there is only one relevant 4-dimensional scalar for every IR region.

The following analysis of the 4-dimensional scalars is similar to the one after (5.2.18) in §5.2. Consider for example the (3-4) IR region in the 3-4 CMS system ($\vec{p}_3 + \vec{p}_4 = 0$). An explicit representation of the 4-parton momenta p_i in this system is given in (C1-3). The relevant 3-parton momenta $q_1 = p_1$, $q_2 = p_2$ and $q_3 = p_3 + p_4$ are 4-dimensional. All 4-dimensional scalars can be expressed e.g. in terms of $\tilde{p}_4^2$. Thus $\tilde{p}_3 \cdot \tilde{p}_4 = p_4(p_3 + p_4) = \tilde{p}_4^2$, $\tilde{p}_3^2 = \tilde{p}_4^2$, $\tilde{p}_4^2 = p_1 \cdot \tilde{p}_3 = p_1 p_3$ etc..

The n-dimensional angular integration of $\tilde{p}_4^2$ is done in App. C. $\tilde{p}_4^2$ is proportional to s_{34} as (C3) and (C4) show. This means that all $\tilde{p}_4^2$ contributions can be dropped except for the true double pole contributions s_{34}^{-2} as long as one is calculating to $O(y^0)$.

After the angular integration (5.3.3) we still have to do the

¹In particular $p_i^2 = 0$ is now to be replaced by $\tilde{p}_i^2 \neq 0$.

invariant mass integration (5.3.1). As already mentioned, to $O(y^0)$ this integration is given simply by (5.3.4).

Integration of the p.v. squared matrix elements

The integration of class A, B and C_I (class notation as in the p.c. case) is similar to the p.c. integration. As discussed in §4.3 there are symmetry relations obeyed by the 4-parton squared matrix elements under $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ exchange which allows one to consider only the (1-3) and (3-4) phase space regions in detail. Care has to be taken in that the p.v. squared matrix elements are effectively² antisymmetric under $1 \leftrightarrow 2$ exchange as compared to the symmetry in the p.c. case due to the replacement of a vector current by an axial vector current. Since there are no true double pole singularities in the diagrams of class A, B and C_I , no care has to be taken to tag 4-dimensional scalars for these contributions. We obtain the same integrand as in the p.c. case

$$(5.3.6) \quad \frac{H^{(3)} p \cdot v \cdot (q_i; A, B, C_I)_{\text{real}}}{C^4 g N_C C_F \{C_F I_C + N_C I_I\}^4 B^{(3)} p \cdot v \cdot (q_i)_{\mu\nu}}$$

where I_C was defined in (4.5.3), I_I in (4.5.7) and C in (4.2.20). $B_{\mu\nu}^{(3)} p \cdot v \cdot$ refers to the p.v. 3-parton Born $O(\alpha_s^-)$ hadron tensor given in (5.2.14). Note that there is no term $R_{\mu\nu}$ as in

(5.2.13) since the momenta q_i are external (i.e. 4-dimensional) momenta for the process considered here, whereas the q_i appearing in §5.2 are to be integrated over and thus are n-dimensional.

²See the results (5.3.10). We find that the anomalous pieces violating the C-symmetry vanish after the angular integrations.

Equation (5.3.6) shows that the p.v. contributions of class A, B and C_I , when evaluated at the $O(y^0)$ level, exhibit the same factorization into Born term and Altarelli-Parisi type kernel (contained in I_C and I_I) as the p.c. contributions.

The contribution from the QCD-QCD graphs in class C denoted by C_G is more subtle due to the occurrence of the double pole singularity s_{34}^{-2} . One has to be careful in performing the limit $y_{34} \rightarrow 0$. As in the p.c. case we have to use the exact formula for the scalar product of 4-parton momenta and therefore get an explicit θ_{1-} dependence. We even find terms where a double pole is left over. These are due to contributions coming from terms proportional to the 4-dimensional scalar $\tilde{p}_4^2$ as explained above.

After performing the θ_{1-} integration we obtain

(5.3.7) $H_{\mu\nu}^{(3)} p.v. (q_1; C_G) \text{ real} = C g^4 N_C^2 \left\{ I_G 4B_{\mu\nu}^{(3)} p.v. (q_1) + (I_R)_{\mu\nu} \right\}$
for those terms which do not contain the scalar $\tilde{p}_4^2$. Here I_G was given in (4.5.8) and

$$(5.3.8) (I_R)_{\mu\nu} = \left[\frac{1}{t_{13}} \epsilon(\mu\nu q q_1) - \frac{1}{t_{23}} \epsilon(\mu\nu q q_2) + \frac{1-t_{12}}{t_{13} t_{23}} \epsilon(\mu\nu q_1 q_2) \right] \\ \cdot \frac{1}{2} \left(1 + \frac{\pi^2}{3} \epsilon \right) \frac{r(1-2\epsilon)}{r^2(1-\epsilon)} \int_0^y dv_{34} \frac{y_{34}^{-\epsilon}}{r^{1-\epsilon}} \int_0^1 dv v^{-\epsilon} (1-v)^{-\epsilon}$$

$$\left[\epsilon v(1-v) - \frac{(1-\epsilon)}{N_{\theta_1}} \int_0^\pi d\theta_1 \sin^{-2\epsilon} \theta_1 \frac{1}{N_{\theta_2}} \int_0^\pi d\theta_2 \sin^{-1-2\epsilon} \theta_2 \frac{\tilde{p}_4^2}{s_{34}^2} \right]$$

The normalization factors N_{θ_1} and N_{θ_2} are given in (4.2.18) and in (C6), respectively.

As is evident from (5.3.7) the p.v. triple-gluon contribution does not manifestly factorize into a Born term contribution and an universal Altarelli-Parisi type kernel. However, doing the $\tilde{p}_4^2$

integration, as described³ in App. C, the non factorizing terms in I_R (5.3.8) can be seen to cancel and we remain with a universal factorizing triple-gluon contribution as in the p.c. case. We emphasize that the inclusion of contributions coming from the integration of $\tilde{p}_4^2$ is of crucial importance in obtaining the right result.

We now turn to the contributions of $e^+e^- \rightarrow q\bar{q}q\bar{q}$. Following again the notation of ERT /42/ we find that the contributions of diagrams in class F vanish identically because of their charge conjugation property. Also the contributions from class E vanish as in the p.c. case. For the diagrams of class D it is sufficient to consider the (3-4) singular regions. Since there is a double pole singularity s_{34}^{-2} we obtain a nonvanishing $\tilde{p}_4^2$ -contribution as in the 3-gluon case. A similar calculation gives

$$(5.3.9) \quad H_{\mu\nu}^{(3)} p.v. (q_1; D) \text{ real} = C g^4 N_C^2 C_F^2 T_R \left\{ I_D 4B_{\mu\nu}^{(3)} p.v. (q_1) + \frac{1}{1-\epsilon} (I_R)_{\mu\nu} \right\}$$

where I_D is given in (4.5.11) and the integral $(I_R)_{\mu\nu}$ containing the $\tilde{p}_4^2$ contribution is identical to the corresponding contribution in the triple gluon case (5.3.8). Since $(I_R)_{\mu\nu}$ is zero we find again the factorization into the Born term contribution and an Altarelli-Parisi type kernel as in the p.c. case (4.5.10).

Integrated 4-parton hadron tensor

Here we present the final results for the real 3-parton

³Note that here is an additional angular dependence compared to the p.c. case.

hadron tensor after integration of the remaining variables y_{ij} and v in the 4-parton hadron tensor integrands. Since the latter ones are identical to those of the p.c. case we can use the p.c. result (4.5.14) and write

$$(5.3.10) \quad H_{\mu\nu}^{(3)p.v.}(q_i)_{\text{real}} = g^2 \left[C_F H_C + N_C H_N/2 + (N_f/3 - 11N_C/6) H_f \right] 4g^2 N_C C_F B_{\mu\nu}^{(3)p.v.}(q_i)$$

where the quantities H_C , H_N and H_f are given in (4.5.13).

We note from (5.3.10) that the $O(y^0)$ IR result factorizes into a Born term contribution and an universal IR factor as in the p.c. case. The universal IR factor is the same for the p.c. and p.v. contributions. Note also that the hadron tensor (5.3.10) contains anomalous pieces of $O(\epsilon^{-1})$ and $O(\epsilon^0)$ due to the anomalous piece in the Born term hadron tensor $B_{\mu\nu}^{(3)p.v.}$ proportional to $\epsilon(\mu\nu q_1 q_2)$, see (5.2.14).

6. THE VIRTUAL CONTRIBUTIONS

6.1 The one loop amplitude for $\gamma, Z \rightarrow q\bar{q}G$ and its renormalization

In this section we briefly recapitulate how to calculate the $O(\alpha_s^{3/2})$ one loop contributions to the amplitude vector current $\rightarrow q\bar{q}G/65/$. We add some comments how UV divergencies can be efficiently regularized and renormalized in the concurrent presence of IR/M singularities.

The virtual $O(\alpha_s^2)$ structure functions and polarization density matrix elements are obtained by folding the $O(g^3)$ one loop amplitude describing $\gamma, Z^0 \rightarrow q\bar{q}G$ with the corresponding $O(g)$ Born term amplitude.

We start this section by discussing some general features of calculating virtual corrections to processes in the framework of perturbative QCD. While calculating the corresponding Feynman diagrams we encounter three types of singularities: ultraviolet (UV), infrared (IR) and mass or collinear (M) singularities. The latter two singularities are collectively called IR/M singularities since we need not distinguish between IR and M singularities in this calculation. Before we can obtain a finite answer these singularities have to be regularized in the intermediate steps of the calculation.

The n- dimensional regularization procedure /56/ turns out to be the most convenient one for a regularization of both types of singularities (UV and IR/M). The reason is that it preserves the Ward identities among regularized Greens functions. Its application in regularization of IR /49,81/ and M /82/ singularities which appear in on shell amplitudes contributing to

mass finite quantities is also far more convenient than the other methods. Other techniques /58/ involve the introduction of regulator masses which render the Feynman integrals more complicated.

Within dimensional regularization a straightforward analysis reveals that in renormalizable field theories the leading UV divergence of a k-loop Feynman graph is given by the pole term $(n-4)^{-k}$ whereas the leading M and IR singularities behave like $(n-4)^{-2k}$. In order to carry out the renormalization program one clearly has to distinguish between the two types of singularities.

Several possibilities are available to perform a higher order loop calculation. We mention Gegenbauer polynomial techniques /70/, Feynman parameter methods /43,57,88,78/ and dispersion methods /84/. If a book-keeping of UV and IR singularities is desired, the dispersion method has some advantages. However, in the latter case one has to be careful not to convert the two types of singularities in intermediate steps of the calculation by using r -function identities like /84/

$$r(n-3-k)r(4-n+k) = (-1)^k r(n-3)r(4-n)$$

where the RHS is independent of k and therefore obliterates the distinction between the various singularities.¹

Each of the above methods has its own technical complexities and one has to decide for each particular loop integral at hand

¹UV divergencies in k-loop graphs show up as $r(\frac{k}{2}(4-n))$ whereas IR/M singularities appear as $r(\frac{k}{2}(n-4))$.

which of the methods is technically advantageous². Sometimes even a mixed application of the above techniques can be recommended.

Before describing how to calculate the one loop amplitude $e^+e^- \rightarrow q\bar{q}G$ we find it useful to make some comments on its renormalization. The n-dimensional method becomes really powerful if one can regularize the UV, M and IR divergences simultaneously and is nevertheless able to perform the renormalization, i.e. can subtract the UV poles (together with finite terms depending on the renormalization scheme).

It is even possible to treat all singularities on the same footing. The distinction between the various types of poles is not needed. In order to subtract the UV poles in a proper way we proceed as follows /84,46/:

1) The first step is to calculate the loop amplitude with all external momenta off-shell ($p_i^2 \neq 0$). In this way all pole terms are of an UV type. Then we choose a specific subtraction scheme ($\overline{MS}$) and make a bookkeeping of the counterterms³.

2) In the second step we compute the loop amplitude with all

²This remark applies to scalar loop integrals and tensorial loop integrals involving loop momentum tensors. In the latter case one has to judiciously decide whether to use one of the above techniques or whether to scalarize /89/ first.

³As it turns out, the UV pole contributions can be immediately be read off in terms of the renormalization constants Z_1' , Z_2 and Z_3' , see (6.1.7).

into a linear combinations of scalar integrals. Inspection of the integrands reveals their UV as well as IR/M behaviour and the UV pole can easily be extracted. However, there is one subtlety.

The scalar two point function

$$(6.1.5) \quad I_2(p) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 (k-p)^2}$$

has no mass scale for $p^2 = 0$. But this integral can be decomposed into the sum of two integrals where one of them is only UV divergent whereas the other one only IR singular. This can be done either by introducing an arbitrary parameter which drops out at the end /98/ or by multiplying the numerator and denominator of the RHS of (5) with $(k \cdot p_i)^2$ where p_i is an arbitrary other external momentum /84/. In this way we can determine the UV pole

$$(6.1.6) \quad (I_2)_{UV} = \frac{i}{16\pi^2} \frac{1}{\epsilon_{UV}} \quad (n \equiv 4 - 2\epsilon)$$

whereas carrying out step two (identification of UV and IR singularities) sets I_2 equal to zero⁴. Therefore we can omit all integrals which only depend on p_i^2 where $p_i^2 = 0$ but have to remember that they yield a counter term given by (6.1.6).

When doing the $O(g^3)$ loop calculation we find that one can determine the UV poles (and thereby the counterterms) without having to do the explicit calculation (i.e. only step two really has to be done). To this end we group all Feynman diagrams as shown in Fig. 14. Note that the diagrams with self-energy insertions at the external legs have to be multiplied by the factor 1/2, see App. D. The first line of Fig. 14 is UV finite due to the Ward identity of the virtual photon (the colour factor

$$4 I_2(p) = \frac{i}{16\pi^2} \left[\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right] \quad \text{for } p^2 = 0.$$

external momenta on shell ($p_i^2 = 0$). Furthermore we neglect the differences between the UV and IR/M pole terms so that we can add them indiscriminately.

This means that in the unrenormalized loop amplitude we cannot distinguish the various kind of singularities. However, having included the counter terms we end up with the renormalized loop amplitude which is UV finite and left over with IR/M poles. We want to emphasize that the two steps do not need two separate calculations as we will show below.

Let us start with the mechanism of counter terms to subtract the UV poles from a (proper) Green's function. To this end we imagine having calculated the Green's function with UV and IR/M divergences kept distinguished:

$$(6.1.1) \quad r^B = \frac{a}{\epsilon_{IR}} + \frac{b}{\epsilon_{IR}} + c + \frac{d}{\epsilon_{UV}}$$

Here B denotes bare or unrenormalized, and $\epsilon_{IR}(\epsilon_{UV})$ symbolize an IR/M(UV) pole ($n \equiv 4 - 2\epsilon$). The MS Green's function follows from the bare one by just omitting the UV pole

$$(6.1.2) \quad r^{MS} = \frac{a}{\epsilon_{IR}} + \frac{b}{\epsilon_{IR}} + c$$

Now assume we perform step one in order to extract the UV pole.

This gives the counter term

$$(6.1.3) \quad r^{ct} = d/\epsilon_{UV}$$

Step two leads us to a Green's function

$$(6.1.4) \quad r^B = \frac{a}{\epsilon_{IR}} + \frac{b+d}{\epsilon_{IR}} + c$$

Subtracting now the counter term (6.1.3) (with $\epsilon_{UV} \rightarrow \epsilon_{IR}$) we recover the MS Green's function (6.1.2).

The above mentioned steps need not be done in two separate calculations. This is because we decompose each Feynman integral

is the same in all 4 diagrams). The second line contains the box graphs. Power counting reveals that the corresponding integrals can not get UV divergent. The third line shows the vertex correction of the quark-gluon-quark vertex, which is UV singular. The last two lines give the Feynman diagrams containing the quark and gluon self-energy, respectively. Their UV poles, however, are well known, since they contain just those subdiagrams determining (in one loop) the renormalization constants Z_1 (of the vertex), Z_2 (of the fermion wave function) and Z_3 (of the gluon wave function). Writing the renormalization constants as $Z = 1 + \delta Z$ we immediately read off the UV poles of the one loop amplitude $T_{\mu\alpha}(g^3)$

$$(6.1.7) \quad T_{\mu\alpha}(g^3)_{UV\text{-pole}} = (\delta Z_2 + \frac{1}{2} \delta Z_3 - \delta Z_1) T_{\mu\alpha}(g) \\ = T_{\mu\alpha}(g) \frac{\alpha_s}{\pi} \frac{1}{4} \left[\frac{11}{6} N_C - \frac{1}{3} N_f \right] \frac{1}{\epsilon_{UV}} \quad (n = 4 - 2\epsilon)$$

In the second step we have inserted the MS renormalization

constants (see e.g. /83/) in the Feynman gauge which is used

throughout the calculation. Referring to step two we end up with the following counter term in the MS scheme

$$(6.1.8) \quad T_{\mu\alpha}^{ct} = \frac{\alpha_s}{2\pi} \left[\frac{1}{3} N_f - \frac{11}{6} N_C \right] \frac{1}{\epsilon_{IR}} T_{\mu\alpha}(g)$$

Since we want to present our results in the $\overline{MS}$ scheme, we give the relation between the two schemes. The $\overline{MS}$ result follows immediately from the corresponding MS result by substituting

$$(6.1.9) \quad g^2 \rightarrow g^2 \exp \left[\epsilon (\ln 4\pi - \gamma) \right]$$

where γ is the Euler constant.

The one-loop amplitude

What remains to be done is step two, i.e. the calculation of

the one loop amplitude with identification of UV and IR poles $\epsilon_{UV} \equiv \epsilon_{IR}$. Since the diagrams with self-energy insertions at the external legs (line 4 and 5 in Fig.14) are proportional to the integral (6.1.5) with $p^2=0$ their contributions can be dropped with the identification $\epsilon_{UV} \equiv \epsilon_{IR}$.

We start by considering the vector current amplitude which has been calculated in /65,98/. The details of the calculation can be found in /65,98/. Here we only outline those points relevant for the present discussion. The vector current amplitude can be evaluated by expanding the amplitude into a set of 13 independent covariants built up from external momenta and the metric tensor. After checking on gauge invariance the number of independent covariants reduces to 7 gauge invariant covariants which are best chosen according to the charge conjugation property of the amplitude. We write

$$(6.1.10) \quad T_{\mu\alpha}^V(g^3) = \sum_{k=1}^7 N_k^V C_{\mu\alpha}^{V,k}$$

We do not want to give explicit expressions for the invariants N_k^V or the covariants $C_{\mu\alpha}^{V,k}$. They can be found in /65,98/. We will rather concentrate on how the structure functions follow from the above amplitude. In chapter 7 we will discuss the expressions for the gluon polarization which can be obtained from (6.1.10).

For the present discussion we need only the IR singular pieces of the one loop amplitude. These occur only in N_7^V where $C_{\mu\alpha}^{V,7}$ has the covariance structure of the Born $O(g)$ amplitude

$$(6.1.11) \quad C_{\mu\alpha}^{V,7} = \gamma \frac{\gamma q_2 + \gamma q_3}{\mu t_{23}} \gamma_\alpha - \gamma \frac{\gamma q_1 + \gamma q_3}{\alpha t_{13}} \gamma_\mu$$

We also need the corresponding axial vector current Born

amplitude

$$(6.1.12) \quad C_{\mu\alpha}^{A,7} = \gamma_{\mu} \gamma_5 \frac{\gamma_{q_2} + \gamma_{q_3}}{t_{23}} \gamma_{\alpha} - \gamma_{\alpha} \frac{\gamma_{q_1} + \gamma_{q_3}}{t_{13}} \gamma_{\mu} \gamma_5$$

It is clear that $C_{\mu\alpha}^{A,7}$ is not a conserved quantity, i.e.

$q_{\mu} C_{\mu\alpha}^{A,7} \neq 0$, even if the external momenta q_i and the external index μ are 4-dimensional, because of the n-dimensionality of the inner

gluon index α . The above prescription of how to calculate a one

loop amplitude does not include the treatment of spurious UV

anomalies which arise in conjunction with γ_5 and dimensionally

regularized UV singularities. UV anomalies have to be canceled by appropriate fixing of the axial vector vertex renormalization

constant /104,79/. Since the identification of UV and IR poles

obliterates the origin of UV anomalies we postulate the UV chiral invariance relation (6.2.3) and understand the vector vertex

renormalization constant to be chosen such that (6.2.3) holds. This relation is indeed a true UV relation since it should hold

even if there are no IR singularities present. (6.2.3) is thus equivalent to establishing the axial vector vertex renormalization

constant. Only the UV anomalies get canceled and (6.2.3) is clearly much easier to handle in the actual calculation.

6.2 The Virtual Structure Functions

The virtual $O(\alpha_s^2)$ hadron tensors are defined by (compare (2.1.15)), and $g^3(g)$ means up to and including the order $g^3(g)$ where $\alpha_s = g^2/(4\pi)$

$$(6.2.1) \quad \begin{aligned} & H_{\mu\nu}^{(3)p.c.}(q_1, q_2, q_3)_{virt} = H_{\mu\nu}^{(3)VV}(q_i)_{virt} \\ & = \sum_{spins} \left\{ J_{\mu}^V(g^3) J_{\nu}^V(g)^* + J_{\mu}^V(g) J_{\nu}^V(g^3)^* \right\} \\ & = - \left\{ \text{Tr } \gamma_{q_1} \gamma_{\mu} \gamma_{\alpha} \gamma_{q_2} \gamma_{\nu} \gamma_{\alpha} \gamma_{q_3} \right\} + \text{Tr } \gamma_{q_1} \gamma_{\mu} \gamma_{\alpha} \gamma_{q_2} \gamma_{\nu} \gamma_{\alpha} \gamma_{q_3} \end{aligned}$$

and

(6.2.2)

$$\begin{aligned} H_{\mu\nu}^{(3)p.v.}(q_1, q_2, q_3)_{virt} &= \frac{1}{2} \left\{ H_{\mu\nu}^{(3)VA}(q_i)_{virt} + H_{\mu\nu}^{(3)AV}(q_i)_{virt} \right\} \\ &= \frac{1}{2} \sum_{spins} \left\{ J_{\mu}^V(g) J_{\nu}^A(g^3)^* + J_{\mu}^A(g) J_{\nu}^V(g^3)^* + J_{\mu}^A(g^3) J_{\nu}^V(g)^* + J_{\mu}^V(g^3) J_{\nu}^A(g)^* \right\} \\ &= - \frac{1}{2} \left\{ \text{Tr } \gamma_{q_1} \gamma_{\mu} \gamma_{\alpha} \gamma_{q_2} \gamma_{\nu} \gamma_{\alpha} \gamma_{q_3} \right\} + \text{Tr } \gamma_{q_1} \gamma_{\mu} \gamma_{\alpha} \gamma_{q_2} \gamma_{\nu} \gamma_{\alpha} \gamma_{q_3} + \\ &\quad \text{Tr } \gamma_{q_1} \gamma_{\mu} \gamma_{\alpha} \gamma_{q_2} \gamma_{\nu} \gamma_{\alpha} \gamma_{q_3} + \text{Tr } \gamma_{q_1} \gamma_{\mu} \gamma_{\alpha} \gamma_{q_2} \gamma_{\nu} \gamma_{\alpha} \gamma_{q_3} \end{aligned}$$

The contribution for the p.c hadron tensor as well as the terms containing $J_{\mu}^A(g) J_{\nu}^V(g^3)^*$ and $J_{\mu}^V(g^3) J_{\nu}^A(g)^*$ can be evaluated without much difficulty using the vector current one loop results /65,98/ discussed in the previous section. The axial vector

vertex renormalization constant must include appropriate counter terms /104,79/ to cancel spurious UV anomalies such that the following UV chiral invariance relation holds

$$(6.2.3) \quad \sum_{spins} \left\{ J_{\mu}^V(g) J_{\nu}^A(g^3)^* + J_{\mu}^A(g^3) J_{\nu}^V(g)^* \right\} = \sum_{spins} \left\{ J_{\mu}^A(g) J_{\nu}^V(g^3)^* + J_{\mu}^V(g^3) J_{\nu}^A(g)^* \right\}$$

As already mentioned, the IR/M singular pieces of the vector

(axial vector) one loop amplitude are proportional to the vector (axial vector) Born (O(g)) amplitude and thus contribute only to the invariant N_7^V/Λ . The real parts of the IR/M divergent contributions to the virtual p.c. and p.v. hadron tensors are then obtained as

$$\begin{aligned} & H_{\mu\nu}^{(3)p.v.}(q_1; \text{sing})_{\text{virt}} \\ &= \langle \text{Re } N_7^V(\text{sing}) \rangle g^4 (-1) \text{Tr} \left\{ \gamma_{q_1 C}^{V,7} \gamma_{q_2 C}^{\overline{A},7} + \gamma_{q_1 C}^{A,7} \gamma_{q_2 C}^{\overline{V},7} \right\} \\ &= 2g^4 \langle \text{Re } N_7^V(\text{sing}) \rangle 4 B_{\mu\nu}^{(3)p.v.}(q_1) \end{aligned}$$

and

$$(6.2.5) \quad H_{\mu\nu}^{(3)p.c.}(q_1; \text{sing})_{\text{virt}} = 2g^4 \langle \text{Re } N_7^V(\text{sing}) \rangle 4 B_{\mu\nu}^{(3)p.c.}(q_1)$$

where $B_{\mu\nu}^{(3)X}$ ($X=p.c., p.v.$) is the Born (α_s) term amplitude squared given in (4.1.10) and (5.2.14). The colour and flavour space summations are denoted by the symbol $\langle \dots \rangle$. Using the explicit expressions in /65,98/ we obtain

$$(6.2.6) \quad 2g^4 \langle \text{Re } N_7^V(\text{sing}) \rangle = g^4 N_C C_F \left\{ C_F H_V^C(\text{sing}) + \frac{N_C}{2} H_V^N(\text{sing}) + \left[\frac{N_f}{3} - \frac{11N_C}{6} \right] H_V^f \right\}$$

where

$$\begin{aligned} (6.2.7) \quad H_V^C(\text{sing}) &= C \left[-\frac{2}{\epsilon} + \frac{1}{\epsilon} (2 \ln(1-x_3) - 3) \right] \\ H_V^N(\text{sing}) &= C \left[-\frac{2}{\epsilon} + \frac{2}{\epsilon} \ln \frac{(1-x_1)(1-x_2)}{1-x_3} \right] \\ H_V^f(\text{sing}) &= C \frac{1}{\epsilon} \end{aligned}$$

and C defined in (4.2.20), $(q^2 x_i = q^2 z_{jk} = 2q_j q_k)$. We can thus write

$$\begin{aligned} (6.2.8) \quad H_{\mu\nu}^{(3)X}(q_1; \text{sing})_{\text{virt}} &= g^2 \left\{ C_F H_V^C(\text{sing}) + \frac{N_C}{2} H_V^N(\text{sing}) + \left[\frac{N_f}{3} - \frac{11N_C}{6} \right] H_V^f \right\} 4g^2 N_C C_F B_{\mu\nu}^{(3)X}(q_1) \end{aligned}$$

The term $H_V^f(\text{sing})$ in (6.2.7) can be recognized as the counter term of our prescription to calculate the loop amplitude in the $\overline{\text{MS}}$ scheme. This can be seen by starting with the loop counter term in the $\overline{\text{MS}}$ scheme

$$(6.1.8) \quad T_{\mu\alpha}^{\text{ct}} = g^2 \frac{1}{8\pi^2} \left[\frac{N_f}{3} - \frac{11N_C}{6} \right] \frac{1}{2\epsilon} T_{\mu\alpha}(g) \quad (n \equiv 4 - 2\epsilon)$$

The corresponding $\overline{\text{MS}}$ counter term is arrived at by the substitution (6.1.9). Using the definition (4.2.20) of C and keeping only terms up to order ϵ we find the $\overline{\text{MS}}$ counter term for the one loop amplitude

$$(6.2.9) \quad T_{\mu\alpha}^{\text{ct}} = g^2 C \left[\frac{N_f}{3} - \frac{11N_C}{6} \right] \frac{1}{2\epsilon} \left[1 - \epsilon \ln \frac{\mu^2}{2} \right] T_{\mu\alpha}(g)$$

It is usual in e^+e^- annihilation to choose the renormalization point as the cm energy, i.e. to set $\mu^2 \equiv q^2$.

Next we use the fact that the Born amplitude populates only the invariant N_7^V with $N_7^V(g) = g t^a$ where t^a are the generators of the $\text{SU}(N_C)$ gauge group. The contribution of the counter term to N_7^V is thus

$$(6.2.10) \quad N_7^{V,a}(\text{ct}) = g^3 t^a C \left[\frac{N_f}{3} - \frac{11N_C}{6} \right] \frac{1}{2\epsilon}$$

The corresponding p.c. (and similar the p.v.) hadron tensor contribution is finally arrived at via

$$\begin{aligned} (6.2.11) \quad H_{\mu\nu}^{(3)p.c.} &= -2g^4 \text{Tr} t^a t^a C \frac{1}{2\epsilon} \text{Tr} q_{1\mu}^{C,V,7} q_{2\nu\alpha}^{C,V,7} \left[\frac{N_f}{3} - \frac{11N_C}{6} \right] \\ &= g^4 N_C C_F \left[\frac{N_f}{3} - \frac{11N_C}{6} \right] C \frac{1}{\epsilon} B_{\mu\nu}^{(3)p.c.} \end{aligned}$$

From this we find the term $H_V^f(\text{sing}) \equiv H_V^f(\text{total}) = C/\epsilon$ appearing in (6.2.6,7).

Results

Comparing the virtual one loop contributions in (6.2.8) with the tree graph contributions in (4.5.14) and (5.3.10) we see that the IR/M singularities as well as the finite anomalous pieces

present in the p.v. Born hadron tensor $B_{\mu\nu}^{(3)p.v.}$ cancel.

What remains to be done is to list the nonsingular contributions of the virtual hadron tensors. After having cancelled the anomalous terms the p.c. hadron tensor in general depends on 5 structure functions and the p.v. hadron tensor on 4 ones, see (2.3.9-10).

We remind the reader that 3 structure functions H_5 , H_8 and H_9 are imaginary. We will now show that the imaginary part of the loop amplitude does not contribute to the $O(\alpha_s^2)$ structure functions.

We first note that the imaginary part of the one loop amplitude is obtained by expanding

$$(6.2.12) \quad \left[\frac{4\pi\mu^2}{-q^2} \right]^\epsilon = \left[\frac{4\pi\mu^2}{+q^2} \right]^\epsilon \left\{ 1 - \frac{\pi^2}{2}\epsilon^2 - i\pi\epsilon + \dots \right\}$$

It is thus proportional to the singular part of the loop amplitude. Now, the singular part has the Born structure and since the p.c. (p.v.) Born term squared contributes only to $\mu \leftrightarrow \nu$ symmetric (antisymmetric) covariants, the structure functions H_5 (H_8 and H_9) which multiply $\mu \leftrightarrow \nu$ antisymmetric (symmetric) covariants remain unpopulated /65,62/, see (2.3.9-10).

We now give the real parts of the p.c. and p.v. virtual hadron tensors. They contribute to the invariants $H_1, \dots, H_4$ and H_6, H_7 , respectively. To this end we write ($i=1,2,3,4,6,7$)

$$(6.2.13) \quad H_i(\text{nonsing}) = g^4 N_C C_F \left[C_F H_{Vi}^C(\text{nonsing}) + \frac{N_C}{2} H_{Vi}^N(\text{nonsing}) \right]$$

For the 4 p.c. structure functions $H_1, \dots, H_4$ we write

$$(6.2.14) \quad H_{Vi}^C(\text{nonsing}) = \hat{H}_i, \quad H_{Vi}^N(\text{nonsing}) = \tilde{H}_i - \hat{H}_i$$

Since the expressions for $\hat{H}_i$ and $\tilde{H}_i$ /65,98/ are rather long they

are given in tables 3 and 4. There we use for abbreviation the quantities

$$(6.2.15) \quad K \equiv \frac{1}{2\pi^2} = 4C \Big|_{\epsilon=0}$$

and

$$(6.2.16)$$

$$r(x,y) \equiv \ln x \ln y - \ln x \ln(1-x) - \ln y \ln(1-y) - \text{Li}_2(x) - \text{Li}_2(y) + \frac{\pi^2}{6}$$

where

$$(6.2.17) \quad \text{Li}_2^x(x) \equiv - \int_0^x dz \frac{\ln(1-z)}{z}$$

We also use the quantities B_1 and B_2 which are already defined in (4.1.13). B_1 and B_2 are related to the Born structure functions via ($J \equiv 1/64\pi^2$)

$$(6.2.18) \quad H_i^{\text{Born}} = 4g^2 N_C C_F H_i^{\text{Born}} = J^{-1} \frac{\alpha_s}{2\pi} N_C C_F \frac{1}{2} H_i^{\text{Born}}$$

and

$$(6.2.19) \quad H_1^{\text{Born}} = -B_1, \quad H_2^{\text{Born}} = H_3^{\text{Born}} = -B_2, \quad H_4^{\text{Born}} = 0$$

Kramer and Lampe /70/ have independently obtained a set of linearly independent one loop helicity structure functions (the 4 scalars in (4.1.8)) by directly folding the one loop amplitudes with the Born term amplitude. We have checked that the two calculations agree with each other.

For the p.v. structure functions we obtain /64/

$$(6.2.20) \quad H_{Vi}^C(\text{nonsing}) = \frac{K}{(1-x_1)(1-x_2)} \left\{ -2x_1 \left[9 + \ln^2(1-x_3) \right] \right. \\ + \frac{(1-x_3)x_3^2 + (1-x_1) \left[x_1(2-x_3) - x_3(1-x_3) \right]}{x_1 x_3} \\ + 2 \ln(1-x_3) \frac{(1-x_3)x_2^2 + x_3 - x_1}{x_3^2} \\ \left. + \ln(1-x_2) \frac{3(1-x_1)(1-x_2) + 2x_2(1-x_3)}{x_2} \right\}$$

$$\begin{aligned}
& + \ln(1-x_1) \frac{(1-x_2)(1-x_1)(x_3-x_1) - 2(1-x_3)^2}{x_1^2} \\
& + 2r(1-x_3, 1-x_2) \frac{(1-x_3)(x_1-x_3)}{1-x_1} \\
& - 2r(1-x_3, 1-x_1) \frac{2x_1(1-x_2) + x_2(1-x_3)}{1-x_2} \}
\end{aligned}$$

$$(6.2.21) \quad H_{V7}^C(\text{nonsing}) = -H_{V6}^C(\text{nonsing}; 1 \leftrightarrow 2)$$

and

$$\begin{aligned}
(6.2.22) \quad H_{V6}^N(\text{nonsing}) &= \frac{K}{(1-x_2)(1-x_1)} \left\{ -2 \frac{(1-x_3)x_3 + (1-x_1)(2-x_3)}{x_3} \right. \\
&+ 2x_1 \left[2 + \ln^2(1-x_3) - \ln^2(1-x_2) - \ln^2(1-x_1) - 2r(1-x_2, 1-x_1) \right] \\
&+ 2(1-x_2) \ln(1-x_2) - 2(1-x_1) \ln(1-x_1) \frac{x_2}{x_1} \\
&- 2r(1-x_3, 1-x_2) \frac{(1-x_3)(x_1-x_3)}{1-x_1} - 2 \ln(1-x_3) \frac{x_2(1-x_3) + (x_3-x_1)}{x_3} \\
&\left. + 2r(1-x_3, 1-x_1) \frac{2x_1(1-x_2) + x_2(1-x_3)}{1-x_2} \right\}
\end{aligned}$$

$$(6.2.23) \quad H_{V7}^N(\text{nonsing}) = -H_{V6}^N(\text{nonsing}; 1 \leftrightarrow 2)$$

The p.v. Born structure functions are given by (compare

$$(6.2.18) \text{ and } (5.2.16))$$

$$(6.2.24) \quad H_6^{\text{Born}} = \frac{2x_1}{(1-x_1)(1-x_2)} = -H_7^{\text{Born}}(1 \leftrightarrow 2).$$

6.3 Results for the $O(\alpha_s^2)$ 3-jet structure functions

Adding up the real tree contributions from section 4 and 5 and the virtual one loop contributions from section 6 we obtain the order α_s^2 3-jet structure functions. To this we add the order α_s contribution and obtain

$$\begin{aligned}
(6.3.1) \quad H_i(q_1, q_2, q_3; y) &= \\
\frac{1}{2} j^{-1} N_{CF} \left[\frac{\alpha_s}{2\pi} \overline{H}_i^{\text{Born}} \left\{ 1 + \frac{\alpha_s}{2\pi} \left[C_F^U C + \frac{N_C}{2} U + \left(\frac{N_f}{3} - \frac{11}{6} N_C \right) U^f \right] \right\} \right. \\
&\left. + \left[\frac{\alpha_s}{2\pi} \right]^2 \frac{1}{K} \left\{ C_F^{H_{Vi}^C}(\text{nonsing}) + \frac{N_C}{2} H_{Vi}^N(\text{nonsing}) \right\} \right]
\end{aligned}$$

where

$$\begin{aligned}
(6.3.2) \quad U^C &\equiv 7 - 2 \ln^2 y + 4 \ln y \ln(1-x_3) - \ln^2(1-x_3) - 3 \ln y + \frac{\pi^2}{3} \\
U^N &\equiv \frac{4}{3} - \ln^2(1-x_1) - \ln^2(1-x_2) + \ln^2(1-x_3) - 2 \ln^2 y \\
&\quad + 4 \ln y \ln \frac{(1-x_1)(1-x_2)}{1-x_3} + \frac{\pi^2}{3}
\end{aligned}$$

$$U^f \equiv -\frac{5}{3} + \ln y$$

The terms $H_{Vi}^{N/C}(\text{nonsing})$ are given in section 6.2.

7. PRESENTATION OF $O(\alpha_s^2)$ CORRECTIONS
 7.1 Thrust distributions

In this chapter we present results on the order of magnitude of the $O(\alpha_s^2)$ corrections to the various $e^+e^- \rightarrow q\bar{q}G$ cross section. Following Sterman and Weinberg /102/, various authors (see e.g. /71/) have proposed variables for measuring the jet topology which are infrared insensitive. Among those are thrust T , sphericity S and acoplanarity A . Here we will be interested in thrust distributions.

There is an experimental difficulty due to confinement: all partons fragment into hadron jets and only these hadrons are detected experimentally. Of course, life for an experimentalist is much easier if one did not have to reconstruct the flavour origin of each jet, i.e. to decide which of the three jet is the quark, antiquark or gluon jet. This is another aspect which motivates the use of thrust distributions: no flavour tagging is necessary. Instead we will rely on energy ordering: a jet gets identified according to the fact if it is the most, second most or least most energetic one. In the following we will calculate thrust distributions of the various e^+e^- cross sections.

Clearly, the p.v. C-odd cross sections do not contribute to observables which do not distinguish particles from antiparticles. Their measurements require the ability to distinguish quark-jets from antiquark-jets. In §7.5 we will show how the p.v. cross sections can be related to charge asymmetries.

3-parton kinematic

Before we are going to discuss thrust distributions let us briefly

recapitulate the 3-parton kinematics. The corresponding phase space can be divided up in six different regions which will be denoted by $(i-j)$, $i \neq j$ and $i, j=1,2,3$ (see Fig.15). Region $(i-j)$ is defined by /71/

$$(7.1.1) \quad x_i > x_j > x_k \equiv 2 - x_i - x_j$$

Here and in the following i, j, k are such that

$$(7.1.2) \quad i \neq j \neq k \neq i, \quad i, j, k \in \{1, 2, 3\}$$

We now want to relate the event frame to the final state momenta. To this end we choose to work in the helicity frame. The results can easily be translated in the transversity frame using (2.3.14). In the helicity frame the event plane is chosen as the $(x-z)$ plane. Thus in each region $(i-j)$ we choose: $\vec{q}_i$ defines the z -axis, i.e.

$$(7.1.3a) \quad \hat{q}_i = \hat{e}_z,$$

and $\vec{q}_j$ defines the x -axis, i.e. lies in the positive $(x-z)$ -half-plane

$$(7.1.3b) \quad \hat{q}_j = (+\sin\theta_{ij}, 0, \cos\theta_{ij}).$$

Since the three momenta lie in a plane, the third momentum is given by

$$(7.1.3c) \quad \begin{aligned} \hat{q}_k &= -\hat{q}_i - \hat{q}_j \\ &= (-\sin\theta_{ik}, 0, \cos\theta_{ik}) \end{aligned}$$

where

$$(7.1.4) \quad q_l = (E_l, \vec{q}_l) = \frac{\sqrt{q^2}}{2} x_l (1, \hat{q}_l) \quad (l = 1, 2, 3)$$

The angles $\theta_{ij} \in [0, \pi]$ between the partons are defined through

$$(7.1.5) \quad \cos\theta_{ij} = \hat{q}_i \cdot \hat{q}_j = 1 - \frac{2(x_i + x_j - 1)}{x_i x_j}$$

$$\sin\theta_{ij} = \frac{2\sqrt{x}}{x_i x_j} \quad , \quad x = \prod_{i=1}^3 (1 - x_i)$$

In region (i-j) we have the kinematical limits (see Fig.15)

$$(7.1.6) \quad \begin{aligned} \frac{2}{3} \leq x_i &\leq 1 \\ 1 - \frac{x_i}{2} \leq x_j &\leq x_i \\ 2(1 - x_i) \leq x_k &\leq 1 - \frac{x_i}{2} \end{aligned}$$

Calculation of thrust frame distributions

Let us define:

- (a) the momentum $\vec{T}$ of the most energetic jet (with energy T) defines the z-axis
- (b) the momentum $\vec{T}_2$ of the second most energetic jet (with energy T_2) lies in the positive (x-z)- half- plane.

According to which parton is most (second most) energetic, we distinguish six different regions (i-j) of the Dalitz plot - these are just the six regions defined above, i.e. in region (i-j) we have the following identifications

$$(7.1.7) \quad \begin{aligned} T &\equiv \max(x_1, x_2, x_3) = x_i & \vec{T} &= \vec{q}_i \\ T_2 &= x_j & \vec{T}_2 &= \vec{q}_j \end{aligned}$$

We denote by $\sigma_\alpha^{(i,j)}(x_1, x_2)$ the helicity cross section σ_α (2.2.18) calculated in the frame (i-j). Using the identifications (7.1.7) we obtain the thrust cross sections $\sigma_\alpha^{(i,j)}(T, T_2)$ in the respective region (i-j) of the Dalitz plot, e.g.

$$(7.1.8) \quad \sigma_\alpha^{(2,3)}(T, T_2) = \sigma_\alpha^{(2,3)}(2 - T - T_2, T)$$

Integrating over T_2 we find $\sigma_\alpha^{(i,j)}(T)$ as a function of thrust T only:

$$(7.1.9) \quad \sigma_\alpha^{(i,j)}(T) = \int_{1-T/2}^T dT_2 \sigma_\alpha^{(i,j)}(T, T_2)$$

The thrust distributions $\sigma_\alpha(T)$ themselves are then the sum of

these 6 functions (7.1.9).

Of course it is not necessary to calculate all thrust cross sections (7.1.9). Making use of the C- symmetry we have

$$(7.1.10a) \quad \begin{aligned} \sigma_\alpha^{(2,1)}(x_1, x_2) &= \pm \sigma_\alpha^{(1,2)}(x_2, x_1) \\ \sigma_\alpha^{(2,3)}(x_1, x_2) &= \pm \sigma_\alpha^{(1,3)}(x_2, x_1) \\ \sigma_\alpha^{(3,1)}(x_1, x_2) &= \pm \sigma_\alpha^{(3,2)}(x_2, x_1) \end{aligned}$$

where the "+" sign applies for $\alpha = U, L, T, I$ and the "-" sign

for $\alpha = 6, 7$. Then we obtain

$$(7.1.10b) \quad \begin{aligned} \sigma_\alpha^{(2,1)}(T) &= \pm \sigma_\alpha^{(1,2)}(T) \\ \sigma_\alpha^{(2,3)}(T) &= \pm \sigma_\alpha^{(1,3)}(T) \\ \sigma_\alpha^{(3,1)}(T) &= \pm \sigma_\alpha^{(3,2)}(T) \end{aligned}$$

In addition we notice that the cross sections σ_α

($\alpha = U, L, T, 6$) are invariant under the reflection of the x- axis

($Ox \rightarrow -Ox$) or equivalently under the reflection $N \rightarrow N + \pi$

whereas $\sigma_{I,7} \rightarrow -\sigma_{I,7}$ under the above change. This can be read off from the angular distribution of the cross section (2.4.4,10),

where the σ_α ($\alpha = U, L, T, 6$) have a N-dependence (if at all) of the form $\cos 2N$ or $\sin 2N$, whereas σ_I and σ_7 depend on N via $\cos N$ and/or $\sin N$. Therefore we can relate two further cross sections

$$(7.1.11) \quad \sigma_\alpha^{(1,3)}(x_1, x_2) = \begin{cases} + \sigma_\alpha^{(1,2)}(x_1, x_2) & \alpha = U, L, T, 6 \\ - \sigma_\alpha^{(1,2)}(x_1, x_2) & \alpha = I, 7 \end{cases}$$

and obtain after a suitable shift of integration

$$(7.1.12) \quad \sigma_\alpha^{(1,3)}(T) = \int_{1-T/2}^T dT_2 \sigma_\alpha^{(1,3)}(T, T_2)$$

$$(\alpha = U, L, T, 6) = + \int_{1-T/2}^T dT_2 \sigma_\alpha^{(1,2)}(T, T_2)$$

$$(\alpha = I, 7) = - \int_{1-T/2}^T dT_2 \sigma_\alpha^{(1,2)}(T, T_2)$$

A similar relation holds for $(3,1) \leftrightarrow (3,2)$.

The total thrust distributions $\sigma_\alpha(T)$ are given as the sum

of the thrust distributions in each region:

$$(7.1.13a) \quad \sigma_\alpha(T) = \sum_{i \neq j} \sigma_\alpha^{(i,j)}(T)$$

With the above symmetry considerations we obtain

(7.1.13b)

$$\begin{aligned} \sigma_{U,L,T}(T) &= \int_{2(1-T)}^T dx_2 \left\{ 2 \sigma_{U,L,T}^{(1,2)}(T, x_2) + \sigma_{U,L,T}^{(3,1)}(2-T-x_2, x_2) \right\} \\ \sigma_I(T) &= \left\{ \int_{1-T/2}^T - \int_{2(1-T)}^{1-T/2} \right\} dx_2 [2\sigma_I^{(1,2)}(T, x_2) - \sigma_I^{(3,1)}(2-T-x_2, x_2)] \\ \sigma_{6,7}(T) &= 0 \end{aligned}$$

Here we can see explicitly that there is no contribution from the p.v. C-odd cross sections. Since the p.v. ϵ -terms in (2.4.8) and (2.4.10) vanish the only way to see the contribution of Z^0 exchange in the angular distribution of the thrust axis is through

G_{B1}/G_{A1} and G_{C1}/G_{A1} in (2.4.8) for transversally polarized beams. In order to get nonvanishing contributions to the cross sections σ_6 and σ_7 (which are odd under $q \leftrightarrow \bar{q}$), one could either consider distributions to which q and $\bar{q}$ contribute differently (e.g. by studying semi-inclusive hadron production, i.e. by weighting q 's and $\bar{q}$'s by their fragmentation functions) or one would have to resort to explicit flavour tagging. This last case will be discussed in section 7.5.

7.2 The p.c. helicity cross sections

Before we can actually calculate the p.c. thrust distributions $\sigma_\alpha(T)$ (7.1.13) we must use relation (2.3.22) to calculate the helicity cross sections $\sigma_\alpha^{(1,2)}(x_1, x_2)$ and $\sigma_\alpha^{(3,1)}(x_1, x_2)$ from the structure functions H_i (6.3.1).

For completeness we present also the relations between the projections of the p.c. hadron tensor and the helicity cross sections in the phase space regions (1-2) and (3-1). The projections $S^{(i)}$ are defined in (4.1.8). In the (1-2) frame one finds

$$\begin{aligned} (7.2.1) \quad J^{-1}\sigma_U &= S^{(1)} - S^{(2)} \\ J^{-1}\sigma_L &= S^{(2)} \\ J^{-1}\sigma_T &= \frac{1}{2}(S^{(1)} + S^{(2)}) + \frac{x_1 x_2 x_3}{4X} (x_1 \cos\theta_{23} S^{(2)} + x_2 \cos\theta_{13} S^{(3)} + x_3 \cos\theta_{12} S^{(4)}) \end{aligned}$$

$J^{-1}\sigma_I = \frac{1}{4\sqrt{2X}} \{ (x_3^2 - x_2^2) S^{(2)} + x_2^2 S^{(3)} - x_3^2 S^{(4)} \}$
The analogous relations in the (3-1) frame can be obtained from the ones above by interchanging $x_1 \leftrightarrow x_3$ and $S^{(2)} \leftrightarrow S^{(4)}$ and by an additional minus sign for σ_I .

The thrust distributions of the Born helicity cross sections can easily be analytically calculated from (2.3.22) or (7.2.1) and (7.1.13). The $O(\alpha_s^2)$ corrections have to be done numerically.

Consider first the $O(\alpha_s)$ results which can be presented in closed form since the above integrations can be done analytically

/40,68/:

(7.2.2)

$$\sigma_{U+L}^{\alpha_s} = \zeta_{2\pi}^{\alpha_s} N_C \left\{ \frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln \frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T} \right\}$$

sections vanish.

Note that in order α_s we have the relation

$$(7.2.3) \quad \sigma_T = \frac{1}{2} \sigma_L$$

The x_i -dependence of the $O(\alpha_s^2)$ cross sections is too complicated to allow for their analytical integrations. The necessary integrations have been done numerically. In the following we will present numerical results for the order α_s^2 corrections to the thrust distribution of the helicity cross sections. We choose to normalize our cross sections by /33/

$$(7.2.4) \quad \sigma = N_C \left\{ 1 + \frac{\alpha_s(q^2)}{\pi} + (1.986 - 0.115n_f) \left[\frac{\alpha_s(q^2)}{\pi} \right]^2 \right\}$$

such that $\xi_{\text{tot}}^f := \xi_{\text{pt}}^f \cdot g_{\text{fl}}^f \cdot A_1$ is the total $O(\alpha_s^2)$ cross section for producing a final state of flavour f . (Note that $1/\xi_{\text{tot}}^f (d\xi^f/d\pi) = 1/\xi (d\xi^{U+L}/d\pi)$).

We have plotted the higher order terms for invariant mass cut-off values $y = 0.04$ and $y = 0.01$. In the PETRA / PEP energy region it is appropriate to take $n_f = 5$ (u, d, s, c, b). Also we take $\alpha_s = 0.16$ which is a realistic value obtained from an analysis of experimental distributions of σ_{U+L} with order α_s^2 corrections included /10/.

We see that the order α_s^2 corrections to the thrust distributions σ_L and σ_T are quite small for $y = 0.04$. In average they amount to an increase of 3% if compared to the order α_s curve. The larger deviation above $T = 0.98$, which should be attributed to the 2-jet region, is caused by the cut-off $y = 0.04$. For $y = 0.01$ the corrections are negative and large. But $y = 0.01$ lies outside the perturbative region as was already

$$\begin{aligned} \sigma_L &= \frac{\alpha_s}{2\pi} N_C \left\{ \frac{2(8T-3T^2-4)}{T^2} \right. \\ \sigma_T &= \frac{1}{2} \sigma_L \\ \sigma_I &= \frac{\alpha_s}{2\pi} N_C \left\{ \sqrt{T(T^2-2T+2)} \left\{ \frac{2}{T^2} \sqrt{T-T} - \frac{1}{T\sqrt{1-T}} \right\} \right. \end{aligned}$$

To obtain the thrust distribution for σ_U it is most

convenient to calculate σ_U from $\sigma_U = (\sigma_U + \sigma_L) - \sigma_L \equiv \sigma_{U+L} - \sigma_L$. The Born thrust cross sections are plotted¹ in Figs. 16-20.

We see that σ_{U+L} and σ_U diverge for $T = 1$ and decrease strongly with decreasing T . The singularity at $T = 1$ has its origin in the singular behaviour of the twice differential structure function H_1 in the IR/M region $x_{1 \rightarrow 1}$ and/or $x_{2 \rightarrow 1}$. σ_I is also singular at $T = 1$ but the singularity is integrable as (7.2.2) shows. Clearly the $O(\alpha_s)$ results are reliable only for thrust values away from $T=1$. The behaviour of the thrust distributions close to the IR/M region could of course be improved by the familiar exponentiation of the IR/M factors (see e.g.

/71/). We will not further pursue this possibility. IR/M cuts are provided by the limits: $x_1, x_2 \leq x_0 \ll 1$ for the variables x_1 and x_2 and similarly for $T: T \leq T_0 \ll 1$ (see discussion later on). It is clear that it makes much more sense to apply the thrust distributions in a kinematical region away from the IR/M singular point $T = 1$. The region $T \leq T_0$ can be considered to be the genuine 3-jet region if $T_0 \approx 0.9$. The integral over this region is the integrated genuine 3-jet cross section. Of course, this cross section depends on how the boundary is defined ($T \leq T_0$ or in other ways). At the kinematic boundary $T = 2/3$ all thrust cross

¹ The normalization constant σ is defined in (7.2.4).

observed in connection with $\sigma_{U+L}/71\%$. For comparison and to show the dependence on the cut-off value y we have also plotted all distributions² for $y = 0.01$. The corrections due to the order α_s^2 terms to the thrust distributions σ_{U+L} and σ_U are somewhat larger, near 40% in average.

The largest order α_s^2 correction is the one to σ_T , about 46% in average.

7.3 A resolution insensitive measure of $O(\alpha_s^2)$ corrections to oriented 3-jet events in $e^+e^-q\bar{q}G$

Let us consider the thrust distribution

$$(7.3.1) \quad \sigma_{2T-L} \equiv 2\sigma_T - 2\sigma_L$$

To order α_s this combination vanishes, see (7.2.3). The corresponding order α_s^2 distribution is shown in Fig.21.

As can be deduced from the discussion after (7.1.10b) the double differential cross section $\sigma_{2T-L}(x_1, x_2)$ is independent of the orientation of the z -axis in the (x, z) helicity plane. In addition σ_{2T-L} does not¹ depend on the cut-off y .

The question arises whether this resolution free measure of the order α_s^2 correction to $e^+e^- \rightarrow q\bar{q}G$ is suited to experimental measurements. The fact that σ_{2T-L} exhibits no azimuthal dependence in the helicity frame suggests that we turn to the transversity frame for a measurement of this quantity. As was mentioned in section 2.3 transversity frame distributions follow easily from the helicity frame distributions by means of (2.3.14).

For σ_{2T-L} the correspondence (2.3.14) reads

$$(7.3.2) \quad \sigma_{2T-L} = \bar{\sigma}_L - \bar{\sigma}_U$$

where $\bar{\sigma}_\alpha$ are the cross sections in the transversity frame.

Referring to the transversity angular distribution (2.4.12) we find

¹The reason is that the cut-off dependent real contributions have the order α_s structure and therefore vanish. Thus, σ_{2T-L} is a clean virtual effect.

²Comments on results for larger cut-off values y are given at the end of section 3.1.

(7.3.3)

$$\frac{d^2\bar{\sigma}}{d\tau d\cos\bar{\theta}} = C_0 \left\{ (\bar{\sigma}_U + \bar{\sigma}_L)(3 - \cos^2\bar{\theta}) + (\bar{\sigma}_L - \bar{\sigma}_U)(1 - 3\cos^2\bar{\theta}) \right\}$$

with some constant C_0 . From Figs. 16 and 21 one can read off that

$$(7.3.4) \quad \left| \frac{\bar{\sigma}_L - \bar{\sigma}_U}{\bar{\sigma}_L + \bar{\sigma}_U} \right| \approx 33\%$$

for most of the thrust range. Since the existence of $\bar{\sigma}_U - \bar{\sigma}_L$ is a pure $O(\alpha_s^2)$ effect, it would be very important to experimentally

verify the predicted smallness of the (resolution free)

$$O(\alpha_s^2) (1 - 3\cos^2\bar{\theta}) - \text{dependence.}$$

7.4 An example for the usefulness of beam polarization

All cross sections can be determined from measurements of angular distributions of the (general) cross section (2.4.4).

Therefore, polarization of the incoming beam does not give any new information in addition to what can already be obtained without polarization (exception: longitudinal polarization is required in order to measure p.v. contributions on top of the Z^0). This fact is well known and is true for all final states /14/.

Nevertheless, with polarized beams it is much easier to measure the coefficients (i.e. the differential cross sections) of the general angular distribution. As an example we show how transverse polarization simplifies the measurement of σ_U and σ_L in particular on top of the Z^0 . This is analogous to the measurement of the jet axis angular distribution at low energies with the SPEAR ring /55/.

We consider the $(\varphi - \theta)$ distribution of the cross section (2.4.8) for the case that the beam is transversely polarized with polarization P_+ and P_- for the e^+ and e^- beam, respectively. Then we have:

$$(7.4.1) \quad h^\pm = \xi_{x'}^\pm = 0, \quad \xi_{y'}^\pm = P_\pm$$

for the polarization vector ξ' in the beam frame. The couplings appearing in (2.4.8) reduce to

(7.4.2)

$$G_{A1} = 1, \quad G_{B1} = -P_- P_+ A_{21}, \quad G_{C1} = -P_- P_+ A_{31}, \quad G_{D4} = A_{44}$$

The $(\theta - \varphi)$ -distribution of the cross section is now given by

$$(7.4.3) \quad \frac{2\pi}{g_{11}^2 \sigma_{pt}} \frac{d^4\sigma}{d\varphi d\cos\theta dx_1 dx_2} = \frac{3}{8}(\alpha_U + 2\alpha_L) \left\{ 1 + \alpha \cos^2\theta - \alpha P_- P_+ \sin^2\theta [A_{21} \cos 2\varphi + A_{31} \sin 2\varphi] + \epsilon A_{44} 2\cos\theta \right\}$$

Here α and ϵ are defined in (2.4.9). Referring to Figs. 5-7 we see that on top of the Z^0 -resonance the strength A_{31} and A_{44} are much smaller than A_{21} which is about (-0.9) (the same holds for very low cms energies $\approx 6\text{GeV}$ as used by SPEAR, here $A_{21} \approx +0.9$). Neglecting therefore the last two terms in (7.4.3) and considering the corresponding thrust distribution we get

$$(7.4.4) \quad \frac{2\pi}{f} \frac{\int \frac{d^3 f}{g_{11} \sigma_{\text{pt}} d\mathbf{p} d\cos\theta d\mathbf{T}}}{\int \frac{d^3 f}{g_{11} \sigma_{\text{pt}} d\mathbf{p} d\cos\theta d\mathbf{T}}} = \frac{3}{g(\sigma_U + 2\sigma_L)} \left\{ 1 + \alpha \cos^2\theta - \alpha A_{21} P_+ \sin^2\theta \cos 2\varphi \right\}$$

Since $\alpha = (\sigma_U - 2\sigma_L) / (\sigma_U + 2\sigma_L)$, measurements of the φ -distribution (averaged over θ) allows one to disentangle σ_U and σ_L . Similar relations can be obtained for the other p.c. cross sections.

By varying P_+ , the coefficient α can be determined even for a fixed polar angle θ which is advantageous if the detector does not have a full polar angle coverage.

7.5 Charge asymmetries in $e^+e^- \rightarrow \gamma, Z^0 \rightarrow q\bar{q}$

The parity violating C-odd parts of the cross section are described up to order α_s^2 by merely two p.v. cross sections $\sigma_6^{\text{p.v.}}$ and $\sigma_7^{\text{p.v.}}$ and depend solely on one electro weak, polarization dependent strength G_{D4} / G_{A1} (see 2.4.4). The cross sections $\sigma_6^{\text{p.v.}}$ and $\sigma_7^{\text{p.v.}}$ correspond to coefficients of certain angular distributions and can therefore be determined by an angular analysis. The angular analysis is usually done by integrating over the two half-spheres leading to angular asymmetries¹.

Since the p.v. parts arise from C-odd V-A interference, they do not contribute to observables which do not distinguish particles from antiparticles, i.e. experimentally, p.v. measurements require the ability to distinguish quarks from antiquarks. Several methods have been suggested [77,95,96,63]. A promising method has been suggested for flavour tagging of b-quark jets. Since the b-mesons have a very long lifetime it is anticipated that one can tag the flavour of b-quark jets with the help of micro vertex detectors.

The p.v. thrust distribution

In order to see the amount of the p.v. corrections alone let us first discuss the p.v. cross sections. The $O(\alpha_s^2)$ corrections to the charge asymmetries will be discussed later on.

Since we need to distinguish between quarks and antiquarks we will consider thrust distributions for the case that e.g. the

¹ historically also called charge asymmetries.

quark is the most energetic particle, i.e. defines the event z-axis. Since σ_6 is a purely polar cross section it does not depend on the direction of the second most energetic particle.

Therefore the thrust distribution $\sigma_6(T)$, given by integration over the phase space regions (1-2) and (1-3) (section 7.1), is simply

$$(7.5.1) \quad \sigma_6(T) = \int_{2(1-T)}^T dx_2 \sigma_6^{(1,2)}(x_1=T, x_2)$$

The cross section σ_7 also shows azimuthal dependence.

Therefore it requires the definition of the x-axis, i.e. one must know the direction of a second jet. We can define two possible distributions:

(a) the antiquark (i.e. $\vec{d}_2$) defines the second axis (in region (1-2) and (1-3))

(b) the second most energetic particle defines the second axis (thrust distribution)

While in case (a) the quark and the antiquark jet have to be detected, case (b) clearly has the advantage that it is not necessary to tag also the antiquark. The respective distribution are obtained via

$$(7.5.2) \quad \sigma_7^a(T) = \int_{2(1-T)}^T dx_2 \sigma_7^{(1,2)}(x_1=T, x_2)$$

$$(7.5.3) \quad \sigma_7^b(T) = \left\{ \int_{1-T/2}^T - \int_{2(1-T)}^{1-T/2} dx_2 \sigma_7^{(1,2)}(x_1=T, x_2) \right\}$$

Using the relations between the cross sections σ_6 , σ_7 and the p.v. structure functions H_6 , H_7 given in (2.3.23) we obtain for the

Born terms

$$(7.5.4) \quad \sigma_6(T) = \frac{\alpha_s}{2\pi} N_C \frac{C_F}{1-T} \left\{ (1+T^2) \ln \frac{2T-1}{1-T} - \frac{3}{2}T^2 + 4T - 8 + \frac{4}{T} \right\}$$

(7.5.5)

$$\sigma_7^a = \frac{\alpha_s}{2\pi} N_C \frac{-C_F}{2\sqrt{T} \sqrt{1-T}} \left\{ (2-3T) \sqrt{1-T} \sqrt{2T-1} + \frac{1}{2}T(4-T) \arcsin \frac{3T-2}{T} \right\}$$

$$(7.5.6) \quad \sigma_7^b = \frac{\alpha_s}{2\pi} N_C \frac{-C_F}{\sqrt{2T}} \left(\frac{T}{\sqrt{1-T}} - 2 \sqrt{2T-1} \right)$$

The thrust distributions for the three cases mentioned are plotted in Figures 22, 23 and 24, respectively.

The other two curves in each figure give the respective $O(\alpha_s^2)$ corrections for cut values $y = 0.04$ and $y = 0.01$, which were obtained by numerical integration. Here again we have taken $n_f = 5$ and $\alpha_s = 0.16$. We see that for $y = 0.04$ the order α_s^2 corrections to σ_6 enlarge the Born term by as much as 50%. This differs from what was found for the p.c. cross sections.

For $y = 0.01$ the order α_s^2 corrections make σ_6 smaller as to be expected from the $\ln^2 y$ term in (6.3.2). Here the corrections are even larger. But $y = 0.01$ lies outside the perturbative region as was already observed in connection with the p.c.

distributions. It is also generally agreed upon that cut values as low as $y = 0.01$ are in a region where fragmentation effects make a separation of jets impossible and therefore should not be used /71/.

The situation is similar for the cross sections σ_7^a and σ_7^b (see Figs. 23, 24). We see that the $y = 0.04$ curve is again appreciably bigger than the $O(\alpha_s)$ curve. The $y = 0.01$ distribution for σ_7^b almost coincides with the respective Born result.

In total we observe that the $y = 0.04$ order α_s^2 corrections for the p.v. cross sections are somewhat larger than those obtained for $\sigma_{U,L,T}$. Since the $O(\alpha_s^2)$ contributions from the real

tree graph contributions are universal (at $O(y^0)$) we trace back this difference to a difference in the contributions of the virtual $O(\alpha_s^2)$ corrections.

Charge asymmetries

We now present the $O(\alpha_s^2)$ results for the charge asymmetries. For the $O(\alpha_s)$ terms this was done earlier by /77/. The $O(\alpha_s^2)$ results presented in the following are new. Before drawing the respective distributions let us briefly recapitulate the definitions of the angular asymmetries.

One angular asymmetry one can define is the forward-backward asymmetry A_θ of the event z-axis (e.g. the thrust-axis) with respect to the electron beam. It follows from the (θ, φ) -distribution (2.4.8):

$$(7.5.7) \quad A_\theta = \frac{3}{2} \langle \cos \theta \rangle$$

$$= \frac{\int_{\theta \leq \pi/2} d\cos\theta - \int_{\theta > \pi/2} d\cos\theta}{\int_{\theta \leq \pi/2} d\cos\theta + \int_{\theta > \pi/2} d\cos\theta} = \frac{\int_{\theta \leq \pi/2} d^3\sigma}{\int_{\theta \leq \pi/2} d\cos\theta \, dx_1 \, dx_2}$$

$$= \frac{3}{4} \frac{G_{D4} \sigma_6^{p.v.}}{G_{A1} \sigma_{U+L}} = 2 \frac{G_{D4}}{G_{A1}} a_\theta$$

where the denominator is (apart from a factor) the angle integrated cross section

$$(7.5.8) \quad \frac{1}{g_{11}^f \sigma_{pt}} \frac{d^2\sigma^f}{dx_1 \, dx_2} = G_{A1}^f \sigma_{U+L} = -J \, G_{A1}^f \, H^{p.c.\mu} \, \mu$$

The other asymmetry is the azimuthal forward-backward asymmetry with respect to the component of the e^+ momentum orthogonal to the event z-axis:

$$(7.5.9) \quad A_\chi = \frac{\int_{\chi \leq \pi/2} d\chi - \int_{\chi > \pi/2} d\chi}{\int_{\chi \leq \pi/2} d\chi + \int_{\chi > \pi/2} d\chi} = \frac{d^4 f(\theta = \frac{\pi}{2})}{d\chi \, d\cos\theta \, dx_1 \, dx_2}$$

$$= - \frac{8\sqrt{2}}{\pi} \frac{G_{D4} \sigma_7^{p.v.}}{G_{A1} (\sigma_U^{p.c.} + 2 \sigma_L^{p.c.})} = 2 \frac{G_{D4}}{G_{A1}} a_\chi$$

Both angular asymmetries are proportional² to the strength G_{D4}/G_{A1} . For a discussion of this factor see section 2.4 (2.4.14-17) and the end of this section.

We present numerical values for the asymmetries a_θ and a_χ for fixed thrust values T , averaged over the energies of the second and third jet. Again we choose the case where the quark defines the z-axis (i.e. is detected). The thrust distributions for the asymmetries can be obtained from the double differential asymmetries given in (7.5.7) and (7.5.9) by simply replacing $\sigma_\alpha(x_1, x_2)$ by $\sigma_\alpha(T)$. For the forward backward asymmetry of the thrust axis we get to lowest order $O(\alpha_s)$ /77/

$$(7.5.10) \quad a_\theta(T) = \frac{3}{8} \frac{(1+T^2) \ln \frac{2T-1}{1-T} + \frac{4}{T} - 8 + 4T - \frac{3}{2} T^2}{(1+T^2) \ln \frac{2T-1}{1-T} - \frac{1}{2} (3T-2)(4-T)}$$

In order to be able to compare our $O(\alpha_s^2)$ corrections with the results of /77/ we will give the azimuthal forward backward asymmetry for the case that the antiquark defines the second event

²The factor 2 in (7.5.7) and (7.5.9) is chosen in order to have the same definitions of a_θ and a_χ as /L:77/. To compare with their formulae one has to substitute: $\sigma^\alpha(L) = g_{11}^{\sigma_{pt}\sigma_\alpha}$ for $\alpha=U, L, T, I$;
 $\sigma^F(L) = g_{44}^{\sigma_{pt}\sigma_6}$; $\sigma^X(L) = 2g_{44}^{\sigma_{pt}\sigma_7}$; $\rho_q(s) = 2g_{44}^{N\sigma_{pt}}$;
 $\sigma_q(s) = g_{11}^{N\sigma_{pt}}$ and therefore $\rho_q/\sigma_q = 2\lambda_{44}$.

In Figs. 29-32 we show the $O(\alpha_s^2)$ corrected charge asymmetry A_θ for u-type quarks as a function of the cm-energy $\sqrt{s}$ for various values of h^+ and h^- where h^+ (h^-) is the helicity of the positron (electron). Finally we plot in Figs. 33-36 the charge asymmetry A_θ on top of the Z^0 -resonance as a function of the helicity h^+ of the positron for various values of the electron helicity h^- . Figs. 33-36 exhibit the strong dependence of the charge asymmetries on the longitudinal beam polarization when the cm-energy equals the Z^0 mass.

axis (corresponds to case (a) in the discussion above). The $O(\alpha_s)$ result is

$$(7.5.11) \quad a_N(T) = \frac{2}{\pi} \frac{(1-T)(2-3T)\sqrt{2T-1} - \frac{1}{2}T(T-4)\sqrt{1-T} \arcsin \frac{3T-2}{T}}{T(1+T)^2 \ln \frac{2T-1}{1-T} + (3T-2)[(1-T) + \frac{1}{2}T(T-4)]}$$

As shown in Fig. 25, $a_\theta(T)$ is a slowly varying function, staying quite close to its end-point value $3/8$ at $T = 1$. $a_N(T)$, on the other hand (see Fig. 26), is maximal at small $T = 2/3$ and falls to zero for $T \rightarrow 1$. This difference is due the fact that σ_Y and σ_6 both diverge for $T \rightarrow 1$ whereas σ_7 remains finite.

The other two curves in Fig. 25 and 26 illustrate that the $O(\alpha_s^2)$ corrections to both functions a_θ and a_N are remarkably small. We observe that the relative big contributions of the order α_s^2 terms to the individual cross sections σ_α almost cancel when inserted in the asymmetry functions a_θ and a_N .

The thrust asymmetry distributions A_θ (7.5.7) and A_N (7.5.9) are the product of the above discussed asymmetry functions a_θ and a_N times the universal strength G_{D4}/G_{A1} . Since the $O(\alpha_s^2)$ corrections to a_θ and a_N are small, A_θ and A_N are nearly the same as those obtained in lowest order. For unpolarized beams the $O(\alpha_s)$ distributions of A_θ and A_N can be found in /77/ (replace $G_{D4}/G_{A1} \rightarrow \frac{1}{2}p/\sigma$ to coincide with their notation). We show in Figs. 27 and 28 the $O(\alpha_s^2)$ corrected charge asymmetries A_θ and A_N for u-type quarks (and for unpolarized beams) as a function of the cm-energy for various values of thrust T .

As we discussed already in §2.4, longitudinal beam polarization is required in order to obtain substantial p.v. contributions on top of the Z^0 . Longitudinal beam polarization is planned to be available for the next generation of e^+e^- machines.

7.6 $O(\alpha_s^2)$ corrections to final state polarization

In the next section we will discuss $O(\alpha_s^2)$ corrections to the polarization of the final state partons. The relevant density matrix elements are obtained from both virtual and real contributions where IR/M singularities cancel among the two contributions. The virtual contributions can be obtained from the $O(\alpha_s^{3/2})$ results of one-loop contributions to the amplitudes describing $e^+e^- \rightarrow q\bar{q}G$ listed in /65,98/. The calculation of the real contributions would in principle require an ab initio calculation of the relevant density matrix elements to the $O(y^0)$ precision aimed for in this work. For the p.v. spin observables this would e.g. require use of the full γ_5 -machinary developed for the calculation of the p.v. cross sections in section 5. However, we feel confident that the universality of the $O(\alpha_s^2)$ real corrections proved for the 6 structure functions (see §4 and §5) $H_1, \dots, H_4, H_6$ and H_7 to $O(y^0)$ implies the same universality for any spin observable. The alluded universality of $O(\alpha_s^2; y^0)$ corrections to all spin observables will not be a simple task to prove in general. Nevertheless, it is very likely to be true and thus will be used as a working hypothesis in the following.

Many final state polarizations could be calculate. Here we choose as illustrative examples the longitudinal polarization of the quark and the linear and circular polarization of the gluon. Lowest order $O(\alpha_s)$ results can be found in /63,86,87/, whereas the $O(\alpha_s^2)$ corrections will be presented here for the first time. As was already mentioned in §2.5, quark- (or antiquark-) spin

density matrices do not contain any information in addition to what is already contained in the helicity cross sections. This manifests itself in the general expressions for the density matrices, see (2.5.6). Thus $O(\alpha_s^2)$ results for the density matrices can be obtained by simply replacing the helicity cross sections appearing in (2.5.6) by their $O(\alpha_s^2)$ values.

For massless quarks the transverse polarization vanishes¹. To calculate the longitudinal quark polarization in the transversity systems we start from the density matrix's polar angle ($\bar{\theta}$ -) distribution normal to the event plane. This distribution can be easily obtained by making in (2.5.7) the substitutions $\alpha \rightarrow \bar{\alpha}$ and $\theta \rightarrow \bar{\theta}$, where $\bar{\alpha}$ is given by (2.3.14) in terms of α and $\bar{\theta}$ is the angle between the electron beam and the normal to the event plane. We obtain

$$(7.6.1) \quad \frac{2}{\sigma_{pt}} \frac{d\bar{\rho}_{\lambda,\lambda'}^q}{dx_1 dx_2 d\cos\bar{\theta}} = (G_{A1} \cdot 1 + G_{A4} \cdot \tau_3) \mathcal{G}_{H1} \\ \frac{3}{g} \left[\sigma_U(1 + \frac{1}{2}\sin^2\bar{\theta}) + \sigma_L(1 + \cos^2\bar{\theta}) + \sigma_T(1 - 3\cos^2\bar{\theta}) \right]$$

Notice that there is no $\cos\bar{\theta}$ - term since up to order α_s^2 the cross section σ_5 is zero.

We find that the degree of longitudinal quark polarization

$$(7.6.2) \quad \bar{P}_Z^q = \frac{\text{Tr}(\rho \cdot \tau_3)}{\text{Tr}\rho} = \frac{G_{A4}}{G_{A1}}$$

is independent of $\bar{\theta}$. This result is the same as in $O(\alpha_s)$, the reason in both cases is the vanishing of σ_5 .

¹ Transverse polarization of the spin 1/2 partons occurs only in the form of transverse spin-correlations between the quark and the antiquark.

Gluon polarization

In this section we evaluate the angle integrated gluon's linear and circular polarization. Let us introduce an abbreviated

notation ($X = p.c., p.v.; \rho, \sigma = \parallel, \perp$)

$$(7.6.3) \quad H^{\lambda\alpha\beta} \equiv H^{\lambda\alpha\beta}_{\mu\nu} g^{\mu\nu}$$

$$(7.6.4) \quad P^X_{\rho\sigma} \equiv \epsilon^\rho_\alpha H^{\lambda\alpha\beta} \epsilon^{\sigma*}_\beta$$

where $\epsilon^\rho_\alpha (\epsilon^{\sigma*}_\beta)$ denotes the polarization vector of the gluon as defined in (2.5.11). $H^{\lambda\alpha\beta}$, defined in (2.5.10), is given to $O(\alpha_s)$ by

$$(7.6.5) \quad H^{p.c.\alpha\beta}_{\mu\nu} = \text{Tr } \gamma q_1 T^\alpha_\mu (\text{Born}) \gamma q_2 \overline{T^\beta_\nu} (\text{Born})$$

$$(7.6.6) \quad H^{p.v.\alpha\beta}_{\mu\nu} = \frac{1}{2} \left\{ \text{Tr } \gamma q_1 T^\alpha_\mu (\text{Born}) \gamma q_2 \overline{T^\beta_\nu} (\text{Born}) \right. \\ \left. + \text{Tr } \gamma q_1 T^\alpha_\mu (\text{Born}) \gamma q_2 \overline{T^\beta_\nu} (\text{Born}) \right\}$$

and it's virtual $O(\alpha_s^2)$ part by

$$(7.6.7) \quad H^{p.c.\alpha\beta}_{\mu\nu} = \text{Tr } \gamma q_1 T^\alpha_\mu (\text{Born}) \gamma q_2 \overline{T^\beta_\nu} (\text{loop})$$

$$+ \text{Tr } \gamma q_1 T^\alpha_\mu (\text{loop}) \gamma q_2 \overline{T^\beta_\nu} (\text{Born})$$

$$(7.6.8) \quad H^{p.v.\alpha\beta}_{\mu\nu} = \text{Tr } \gamma q_1 T^\alpha_\mu (\text{Born}) \gamma q_2 \overline{T^\beta_\nu} (\text{loop})$$

$$+ \text{Tr } \gamma q_1 T^\alpha_\mu (\text{loop}) \gamma q_2 \overline{T^\beta_\nu} (\text{Born})$$

In the last equation we have used the ultraviolet chiral

invariance relation (6.2.3). The cross sections $d\sigma_\zeta$ ($\zeta = +, -, \parallel, \perp$)

appearing in (2.5.8) are then given by

$$(7.6.9) \quad d\sigma_\zeta = \frac{-g_{41}}{64\pi^2} \left\{ G_{A1} \frac{p.c.}{\zeta\zeta} + G_{A4} \frac{p.v.}{\zeta\zeta} \right\}$$

Using the relation

$$(7.6.10) \quad \epsilon^{\pm*}_\beta \epsilon^\pm_\alpha = \frac{1}{2} \left\{ \epsilon^{\parallel\parallel} \epsilon^\pm_\beta \epsilon^\pm_\alpha + \epsilon^{\perp\perp} \epsilon^\pm_\beta \epsilon^\pm_\alpha \pm i(\epsilon^{\parallel\perp} \epsilon^\pm_\beta \epsilon^\pm_\alpha - \epsilon^{\perp\parallel} \epsilon^\pm_\beta \epsilon^\pm_\alpha) \right\}$$

and defining

$$(7.6.11) \quad Z^X_{\parallel\parallel} \equiv P^X_{\parallel\parallel} - P^X_{\perp\perp}, \quad Z^X_{\text{sum}} \equiv P^X_{\parallel\parallel} + P^X_{\perp\perp}, \\ Z^X_{\text{cir}} \equiv i(P^X_{\parallel\perp} - P^X_{\perp\parallel})$$

we find

$$(7.6.12) \quad d\sigma_{\parallel} - d\sigma_{\perp} = \frac{-g_{41}}{64\pi^2} \left\{ G_{A1} \frac{p.c.}{Z_{\parallel\parallel}} + G_{A4} \frac{p.v.}{Z_{\parallel\parallel}} \right\}$$

$$(7.6.13) \quad d\sigma_+ + d\sigma_- = d\sigma_{\parallel} + d\sigma_{\perp} \\ = \frac{-g_{41}}{64\pi^2} \left\{ G_{A1} \frac{p.c.}{Z_{\text{sum}}} + G_{A4} \frac{p.v.}{Z_{\text{sum}}} \right\}$$

$$(7.6.14)$$

$$d\sigma_+ - d\sigma_- = \frac{-i g_{41}}{64\pi^2} \left(\epsilon^{\parallel\parallel} \epsilon^{\perp\perp} - \epsilon^{\perp\perp} \epsilon^{\parallel\parallel} \right) \left\{ G_{A1} \frac{p.c.}{Z_{\text{lin}}} + G_{A4} \frac{p.v.}{Z_{\text{cir}}} \right\} \\ = \frac{-g_{41}}{64\pi^2} \left\{ G_{A1} \frac{p.c.}{Z_{\text{cir}}} + G_{A4} \frac{p.v.}{Z_{\text{cir}}} \right\}$$

So the task is to evaluate the quantities $Z^X_{\text{lin}}, Z^X_{\text{sum}}$ and Z^X_{cir} .

The virtual contributions to the above quantities can be evaluated without much difficulty using the results /65/ for the one loop amplitude. For the real or tree part we assume that $Z^X_{\text{lin,sum,cir}}$ factorize into the respective Born terms and the universal IR/M factors in much the same way as the contributions to the real structure functions. Clearly, an explicit analysis would require care with γ_5 as was necessary in the calculation of the p.v. structure functions.

We find again that the IR/M poles cancel against virtual and real contributions. Furthermore, only the real part of the one loop amplitude does contribute, or equivalently

$$(7.6.15) \quad Z^{p.c.}_{\text{cir}} = Z^{p.v.}_{\text{lin}} = Z^{p.v.}_{\text{sum}} = 0$$

We will thus omit the indices p.c. and p.v. in the remaining quantities $Z^{p.c.}_{\text{lin}}, Z^{p.c.}_{\text{sum}}$ and $Z^{p.v.}_{\text{cir}}$. (7.6.13) and (2.5.13-14) imply that

$$(7.6.16) \quad P_{\text{lin}} = \frac{Z_{\text{lin}}}{Z_{\text{sum}}}, \quad P_{\text{cir}} = \frac{G_{A4}}{G_{A1}} \frac{Z_{\text{cir}}}{Z_{\text{sum}}} \equiv - \frac{G_{A4}}{G_{A1}} P_{\text{cir}}$$

i.e. the gluon's linear polarization does not depend on beam polarization and/or neutral current effects whereas the circular polarization does via G_{A4}/G_{A1} . This observation has already been

made for the Born terms in /63,87/. The $O(\alpha_s^2)$ corrections thus do not influence the dependence off the gluon polarization on beam polarization and weak effects.

Before we give the final results let us quote the corresponding expressions for the Born terms. Omitting the overall factor $4g^2 N_C C_F$ the Born terms are given by

$$\begin{aligned}
 (7.6.17) \quad Z_{\text{lin}}(x_1, x_2; \alpha_s) &= -4 \frac{(1-x_3)}{(1-x_1)(1-x_2)} \frac{x_1^2 + x_2^2}{2} \\
 Z_{\text{sum}}(x_1, x_2; \alpha_s) &= -2 \frac{(1-x_3)}{(1-x_1)(1-x_2)} \frac{x_1^2 - x_2^2}{2} \\
 Z_{\text{cir}}(x_1, x_2; \alpha_s) &= 2 \frac{(1-x_3)}{(1-x_1)(1-x_2)} \frac{x_1^2 - x_2^2}{2}
 \end{aligned}$$

For the $O(\alpha_s)$ gluon polarization (7.6.16) we obtain

$$\begin{aligned}
 (7.6.18) \quad P_{\text{lin}}(x_1, x_2) &= 2 \frac{1-x_3}{x_1^2 + x_2^2} \\
 P_{\text{cir}}(x_1, x_2) &= \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2}
 \end{aligned}$$

For the results up to $O(\alpha_s^2)$ (including the $O(\alpha_s)$ results) we obtain ($A = \text{lin, sum, cir}$)

$$\begin{aligned}
 (7.6.19) \quad Z_A(x_1, x_2) &= Z_A(x_1, x_2; \alpha_s) \left\{ 1 + \frac{\alpha_s}{2\pi} \left[C_F(7 - 2\ln^2 y + 4\ln y \ln(1-x_3) - \ln^2(1-x_3) - 3\ln y + \frac{\pi^2}{3}) \right. \right. \\
 &\quad + \frac{N_C}{2} \left(\frac{4}{3} - \ln^2(1-x_2) - \ln^2(1-x_1) + \ln^2(x_1+x_2-1) - 2\ln^2 y \right. \\
 &\quad \left. \left. + 4\ln y \ln \frac{(1-x_1)(1-x_2)}{1-x_3} + \frac{\pi^2}{3} \right) + \left(\frac{N_F}{3} - \frac{11}{6} N_C \right) \left(-\frac{5}{3} + \ln y \right) \right] \Big\} \\
 \alpha_s \left\{ -\frac{N_C}{2\pi} \left[F_A^1(x_1, x_2) + \ln(1-x_2) F_A^2(x_1, x_2) + \ln(1-x_1) F_A^2(x_2, x_1) \right. \right. \\
 &\quad \left. \left. + (\ln^2(1-x_2) + \ln^2(1-x_1)) Z_A(x_1, x_2; \alpha_s) + r(1-x_2, 1-x_1) F_A^3(x_1, x_2) \right] \right. \\
 &\quad \left. + (C_F - N_C/2) \left[F_A^4(x_1, x_2) + \ln(1-x_3) F_A^5(x_1, x_2) - \ln^2(1-x_3) Z_A(x_1, x_2; \alpha_s) \right. \right. \\
 &\quad \left. \left. + \ln(1-x_2) F_A^6(x_1, x_2) + \ln(1-x_1) F_A^6(x_2, x_1) + r(1-x_3, 1-x_2) F_A^7(x_1, x_2) \right. \right. \\
 &\quad \left. \left. + r(1-x_3, 1-x_1) F_A^7(x_2, x_1) \right] \right\}
 \end{aligned}$$

where the functions F_A^i ($i = 1, \dots, 7$) are given in table 7 and $r(x, y)$ is defined in (6.2.16).

Inserting the above expressions in (7.6.16) we get the gluon's linear and circular polarization up to $O(\alpha_s^2)$. Instead of writing down the rather long expressions for the double differential results we present graphically the $O(\alpha_s^2)$ corrections to the gluon's polarization in terms of thrust distributions. To this end we integrate the numerator and denominator of the gluon's polarization in each phase space region over the least energetic energy, in complete analogy to the thrust distributions of the p.c. helicity cross sections discussed in sections 7.1 and 7.2. Using the symmetry of the integrands we can write

$$(7.6.20) \quad P_{\text{lin}}(T) = \frac{Z_{\text{lin}}(T)}{Z_{\text{sum}}(T)}$$

where ($A = \text{lin, sum}$)

$$(7.6.21) \quad Z_A(T) = 2 \left\{ \int_{2(1-T)}^T dx_2 Z_A(T, x_2) + \int_{1-T/2}^T dx_2 Z_A(2-T-x_2, x_2) \right\}$$

To lowest order $O(\alpha_s)$ we obtain /86/

$$(7.6.22) \quad P_{\text{lin}}(T) = 1 - \frac{2T(1-T)\ln\xi + T(3T-2)^2}{2(3T^2-3T+2)\ln\xi - 3T(3T-2)(2-T)}$$

where we have set $\xi \equiv (2T-1)/(1-T)$. It is readily seen that $3/4 \leq P_{\text{lin}}(T) \leq 1$, compare figure 37. This high degree of polarization for the entire thrust range reflects the fact that the region in phase space contributes very little to the cross section where the $O(\alpha_s)$ polarization (7.6.18) vanishes ($x_3 \rightarrow 1$).

In figure 37 we have also plotted the order α_s^2 corrected linear polarization of the gluon. We find that the both curves coincide for cut off values $y = 0.04$ and $y = 0.01$, i.e. the y-dependencies of the numerator and denominator cancel. The

gluon's linear polarization is thus a IR/M stable quantity. We note that the $O(\alpha_s^2)$ double differential distribution of the gluon's linear polarization in (7.6.19) contains an overall factor $(1-x_3)$ just as the $O(\alpha_s)$ distribution (7.6.18). The high degree of polarization found to $O(\alpha_s)$ thus appears in $O(\alpha_s^2)$, too. These strong polarization effects are expected to be observable as asymmetry effects in the gluon jet. For a discussion of possible observable asymmetry effects see e.g. /87/. Clearly, a measurement of the linear polarization of the gluon would be a sensitive, direct test of the vector property of the gluon, providing a test for the validity of QCD.

The (angle integrated) gluon's circular polarization receives its contribution from the p.v., charge conjugation odd hadron tensor (compare (7.6.15,16)). Therefore the circular polarization is only measurable if one distinguishes quarks and antiquarks.² This is in complete analogy to what was said in connection with the p.v. structure functions and the charge asymmetries in section 7.5. To illustrate the order of magnitude of the $O(\alpha_s^2)$ corrections we again choose the case where the quark jet is

²In /87/, H.A. Olsen et al. consider the angular dependent gluon circular polarization to $O(\alpha_s)$. However, they restrict their analysis to the part of the hadron tensor that is symmetric under exchange of quark and antiquark ($q \leftrightarrow \bar{q}$). When integrated over the Euler angles θ and ϕ this part vanishes; compare (5.2) and (5.9) in /87/ (only C-odd quantities survive). The analysis of /87/ can therefore not be used as a basis of the $O(\alpha_s)$ angle-integrated circular polarization considered here.

detected with the remaining energy integrated over. We then obtain the gluon's circular polarization as a function³ of thrust T where the thrust axis is given by the quark momentum:

$$(7.6.23) \quad P_{\text{cir}}(T) = \frac{G_{A4}}{G_{A1}} \frac{Z_{\text{cir}}(T)}{Z_{\text{sum}}(T)} \equiv - \frac{G_{A4}}{G_{A1}} \dot{P}_{\text{cir}}(T)$$

where $(A = \text{cir, sum})$

$$(7.6.24) \quad Z_A(T) = \int_{2(1-T)}^T dx_2 Z_A(T, x_2)$$

To lowest order $O(\alpha_s)$ we obtain ($\xi \equiv (2T-1)/(1-T)$)

$$(7.6.25) \quad \dot{P}_{\text{cir}}(T) = \frac{(3T-2)(4-T) - 2(1-T^2)\ln\xi}{2(1+T^2)\ln\xi - (3T-2)(4-T)}$$

The lowest order distribution together with the α_s^2 corrected curves for $y = 0.04$ and $y = 0.01$ are shown in figure 38. The circular polarization is rather small, $P_{\text{cir}} \leq 1/4$, and both $O(\alpha_s^2)$ distributions are somewhat smaller than the $O(\alpha_s)$ distribution.

³It might be unusual to plot the gluon's circular polarization as a function of the quark energy x_1 , but remember that the C-odd gluon's circular polarization is zero for all values of x_3 when integrated over x_1 and x_2 .

8. SUMMARY AND OUTLOOK

Jets in e^+e^- annihilation are an excellent laboratory for precision studies of the standard theory of electro-weak and strong interactions with the gauge group $SU(3) \times SU(2) \times U(1)_Y$. Up until now only the angular averaged 3-jet cross section has been measured. In the near future there will be much more data coming from PEP, TRISTAN, SLC and LEP which should allow for a determination of the various helicity cross sections and their comparison with theoretical predictions.

In this paper we have presented the general structure of 3-jet events in e^+e^- annihilation including electro-weak and beam polarization effects. We have described in detail the $O(\alpha_s^2)$ corrections to

- i) the complete set of structure functions that describe the space orientation of 3-jet events
- ii) the charge asymmetries in $e^+e^- \rightarrow \gamma, Z \rightarrow 3\text{-jets}$, and
- iii) the polarization of the final state quarks and the gluon.

We found that the $O(\alpha_s^2)$ corrections to the 3-jet helicity cross section formula are important and have to be included for a reliable comparison of theory and experiment. However, the $O(\alpha_s^2)$ corrections to the charge asymmetries and final state polarizations are remarkably small.

We proposed an interesting resolution insensitive measure of $O(\alpha_s^2)$ corrections to oriented 3-jet events in $e^+e^- \rightarrow q\bar{q}G$ which should be measurable already with present data.

These higher order QCD calculations will find extensive application at LEP and SLC energies, where the availability of

beam polarization and improved statistics will enable one to disentangle most of the structure functions. The high energy as well as the high statistics for $Z \rightarrow 3\text{-jets}$ expected for SLC and LEP will allow further detailed quantitative tests of QCD. These include confirmation of the non-abelian nature of QCD, precise determination of α_s and determination of the electro-weak coupling constants of quarks and leptons.

Higher order perturbative QCD calculations require a rather elaborate technical machinery. We hope that this detailed presentation of the many calculational aspects of the problem will be of help for many higher order QCD calculations that have to be done for the interpretation of the experiments at future high-energy machines.

In view of the simple final results of $O(\alpha_s^2)$ tree graph contributions we would like to comment on general features and speculate about possible technical simplifications.

First we note that the leading log terms of all 3-jet structure functions multiply the corresponding Born term structure. This can be traced back to the IR/M behaviour of the real contributions. We observe that to $O(y^0)$ the real tree contributions factorize into the 3-parton Born term and an universal IR/M factor when integrated over unresolved parton pairs. Decomposing the real structure functions according to

$$(8.1) \quad H_1(\alpha_s^2; \text{tree}) \equiv$$

$$g^2 \left\{ C_F H_1^C(\alpha_s^2; \text{tree}) + \frac{N_C}{2} H_1^N(\alpha_s^2; \text{tree}) + \left(\frac{N_f}{3} - \frac{11}{6} N_C \right) H_1^f(\alpha_s^2; \text{tree}) \right\}$$

we find ($A = C, N, f$)

$$(8.2) \quad H_1^A(\alpha_s^2; \text{tree}) = H_A H_1(\alpha_s)$$

where the universal IR/M factors H_A are defined in (4.5.13).

are singular in the 3-jet region but make no contribution in the leading log approximation. They are therefore not related to Altarelli-Parisi kernels. But they are crucial in order to get the right integrated N_C IR/M-factor (i.e. the IR/M-factor whose IR/M poles cancel against the poles of the corresponding virtual contribution). The existence of such singular terms indicates that the leading log approximation may not be used to extract information about subleading logs.

Second, the diagrams involving the triple gluon graphs do not factorize at the unintegrated level. In this case the azimuthal angle type integrations have to be done before one arrives at a factorized form. In the p.c. case this is simply to see whereas in the p.v. case there is yet another azimuthal angle type integration to be done due to the subtleties of γ_5 in n dimensions. It is exactly the linear polarization of the off-shell gluon that induces these azimuthal angular dependences. Only azimuthal averaged gluons exhibit factorization.

Third, we like to caution again that the nonleading terms are effected by precisely how the range of integration is specified in the presence of overlapping singularities.

Fourth, the above factorization has been shown to be true in the $O(y^0)$ approximation. It would be interesting to find out whether such a factorization also occurs for higher order terms in y . Unfortunately, the $O(y^0)$ approximation has been quite crucial in keeping our calculations of the structure functions at a manageable level. A more accurate calculation of $O(\alpha_s^2)$ contributions to the structure functions would pose formidable difficulties and, in particular, would be more difficult than the

The interpretation of (8.2) is the following: when integrated over unresolved regions of phase space the tree contributions factorize into the corresponding Born term structure and an universal (4-3) IR/M-factor which is specific to each independent colour contribution.

In addition we observe that there is a one to one correspondence between the (4-3) IR/M-factor for the QED like 4-parton subprocess (i.e. with colour factor C_F) and the IR/M-factor appearing in the reduction 3-parton down to 2-jets. The (4-3) C_F IR/M-factor H^C (4.5.13) can be obtained from the (3-2) IR/M-factor in (5.2.26) by substituting $q^2 = s_{12}$ in the constant C . Conversely the (3-2) IR/M-factor is the same as the QED type IR/M-factor for $z_{12} \rightarrow 1$. This correspondence already appears in the respective integrands, which can be transformed into each other by interchanging v and $(1-v)$.

It is interesting to note that the above mentioned factorizing of a N -parton process into the $(N-1)$ - Born term parton process and an universal IR/M-factor already appears before integration in all but one case. Even the squared matrix elements exhibit this factorization when evaluated in the limit of unresolved parton pairs. If the diagrams are classified according to their colour factors each class receives its own IR/M-factor. It is not surprising that these unintegrated IR/M-factors are the Altarelli-Parisi kernels apart from the one exception discussed after (4.5.9). Thus, in the reduction $qG \rightarrow q$, $GG \rightarrow G$ and $q\bar{q} \rightarrow G$ we find the kernels $P_{qq}(v)$, $P_{GG}(v)$ and $P_{qG}(v)$, respectively.

However, four comments have to be added. First note that the (1-3) singular contributions proportional to the colour factor N_C

corresponding calculation of the angle-averaged 3-jet cross section /51/.

However, keeping in mind these caveats we would like to speculate on how higher order perturbative calculations could be simplified. Assuming that the three (integrated) IR/M-factors in (8.2) (corresponding to the three colour factors N_C , C_F and $T_R = N_f/2$) are as universal as the Alcarelli-Parisi kernels, we propose the following rule: the contributions of the tree diagrams to jet cross sections are obtained by simply defining the $O(\alpha_s^2)$ contributions as the corresponding $O(\alpha_s)$ Born terms times the universal IR/M-factors.

We would like to add that proofs of such conjectures exist in QED using eikonal techniques. However, these proofs have not yet been extended to QCD due to complications arising from the non-Abelian structure of QCD.

We feel that more insight into the structure of the higher order QCD corrections can be obtained by the use of helicity methods which have been shown to lead to huge simplifications in the calculation of tree graph contributions /19,28,37/. This would amount to a variant of the dimensional reduction scheme. In particular such an approach would ameliorate the γ_5 -problem, since γ_5 would live in 4-dimensional environment and thus one would simply obtain the p.v. helicity amplitudes from the p.c. helicity amplitudes by mere sign changes. One can also hope for a considerable simplification in the loop calculations using helicity methods. These possibilities are under investigation.

One has to mention, though, that there are internal inconsistencies as well as problems with gauge invariance in the

dimensional reduction scheme. We are presently investigating whether helicity methods can be applied to higher order QCD calculations involving UV and IR/M singularities despite of the above difficulties.

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Parts of this thesis have already been published in /64,66/.

$$\begin{aligned}
 \text{(A4)} \quad \text{trace} &= 4\text{-dim.terms} + (n-4)\text{Tr}\gamma_1\gamma_3\gamma_5\gamma_2\gamma_3\gamma_\nu \\
 &\quad \text{(i)} \\
 \text{trace} &= 4\text{-dim.terms} - (n-4)\text{Tr}\gamma_1\gamma_3\gamma_5\gamma_2\gamma_3\gamma_\nu \\
 &\quad \text{(ii)}
 \end{aligned}$$

We find the terms proportional $(n-4)$ differ in sign. Integration of the 3-parton hadron tensor over unresolved parton pairs results in IR/M poles in $(n-4)$. Multiplying these poles with the $(n-4)$ -terms in (A4) we end up with finite terms that are different depending on how the contraction is performed. Clearly, this is unphysical.

Regularization

We start the analysis by briefly recapitulating some well known facts of dimensional regularization. We choose this regularization procedure in order to preserve local gauge invariance. Then UV and IR/M divergences show up as poles in $(n-4)$. Our demands are that both, UV and IR/M singularities and anomalies, get treated by one regularization procedure (another possibility was used by /105/) in the hope that this procedure will prove to be practicable.

There is only one place where one runs into trouble with the extension of the Lorentz covariants. This is where the $\epsilon(\alpha, \beta, \gamma, \delta)$ -tensor comes into play, since it is a genuine 4-dimensional object /25/. This is expressed by its property that in 4 dimensions the value of an arbitrary Lorentz index is equal to exactly one of the values of the indices of ϵ . Correspondingly there are identities involving ϵ depending crucially on 4-dimensionality as

$$\begin{aligned}
 \text{(A5)} \quad &\epsilon(\mu_1\mu_2\mu_3\mu_4)\epsilon(\nu_1\nu_2\nu_3\nu_4) = -\det(g_{\alpha\beta}) \quad \alpha = \mu_i, \quad \beta = \nu_j \\
 &\text{and as a consequence}^1
 \end{aligned}$$

¹This identity is sometimes known as the Schouten-identity.

Appendix A: γ_5 and dimensional regularization

The merge of dimensional regularization and γ_5 is a problematic one. If one does not use a consistent γ_5 -scheme calculations of p.v. quantities involving parity-odd fermion traces will lead to ambiguous results. In this section we discuss various versions of dimensional regularization and analyse their applicability to calculations of p.v. cross sections.

A simple example

Before discussing the general framework we want to give a simple example in order to show how ambiguities arise in an inconsistent scheme. To this end we consider the V-A interference appearing in e^+e^- annihilation into 3 partons in $O(\alpha_s)$. The vector and axial vector amplitudes are defined, respectively, by

$$\begin{aligned}
 \text{(A1)} \quad T_{\mu\alpha}^V &= \gamma_\alpha \frac{\gamma_1 + \gamma_3}{2q_1q_3} \gamma_\mu - \gamma_\mu \frac{\gamma_2 + \gamma_3}{2q_2q_3} \gamma_\alpha \\
 T_{\mu\alpha}^A &= \gamma_\alpha \frac{\gamma_1 + \gamma_3}{2q_1q_3} \gamma_\mu \gamma_5 - \gamma_\mu \gamma_5 \frac{\gamma_2 + \gamma_3}{2q_2q_3} \gamma_\alpha
 \end{aligned}$$

Inserting these expressions into the definition of the V-A hadron tensor (2.1.15) we find

$$\text{(A2)} \quad H_{\mu\nu}^{VA} = \frac{\text{Tr} \gamma_1 \gamma_\alpha (\gamma_1 + \gamma_3) \gamma_\mu \gamma_5 \gamma_2 \gamma^\alpha (\gamma_2 + \gamma_3) \gamma_\nu}{2q_1q_3 2q_2q_3} + \dots$$

where we have left out terms which are irrelevant for the following argument. If one naively assumes

$$\text{(A3)} \quad \left\{ \gamma_\rho, \gamma_5 \right\} = 0$$

one encounters an ambiguity in evaluating the trace (A2).

Depending on the way we perform the α -contraction, either

$$\text{(i)} \quad \gamma_\alpha (\gamma_2 + \gamma_3) \gamma_\nu \gamma_1 \gamma^\alpha, \quad \text{or}$$

$$\text{(ii)} \quad \gamma_\alpha (\gamma_1 + \gamma_3) \gamma_\mu \gamma_5 \gamma_2 \gamma^\alpha$$

we get different results:

$$(A6) \quad g_{\mu_1\nu} \epsilon(\mu_2\mu_3\mu_4\mu_5) + \text{cycl.}(\mu_i) = 0$$

Two possibilities are open:

- a) no use is made of (A5), i.e. there is no possibility to reduce a product of several ϵ -tensors
- b) in addition to the "n-dimensional" covariants also "(n-4)-dimensional" ones (or equivalently 4-dimensional ones) are introduced.

However, (A6) prevents one from finding easily a unique normal form for the covariants. γ_5 leads to the same problems, but there exist the proposal to use dimensional regularization without specifying γ_5 /30,31/. Yet we will see that this fails.

Problems with γ_5

γ_5 is defined for $n = 4$ by:

$$(A7) \quad \text{Tr} \gamma_5 \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4} = \epsilon(\mu_1\mu_2\mu_3\mu_4) \text{ i Tr} 1$$

$$(A8) \quad \left\{ \gamma_{\rho'} \gamma_5 \right\} = 0$$

Within usual dimensional regularization

$$(A9) \quad \gamma_{\rho'} \gamma^{\rho} = n$$

this leads to

$$(A10) \quad n \text{Tr} \gamma_5 = n(n-2) \text{Tr} \gamma_5 \gamma_{\mu_1} \gamma_{\mu_2} = n(n-2)(n-4) \text{Tr} \gamma_5 \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4} = 0$$

Then $\text{Tr} \gamma_5 \gamma_{\alpha} \gamma_{\beta} \gamma_{\gamma} \gamma_{\delta}$ cannot be analytically continued from 4 to $n \neq 4$.

Of course when only even parity fermionic loops are present

the trace $\text{Tr} \gamma_5 \gamma_{\alpha} \gamma_{\beta} \gamma_{\gamma} \gamma_{\delta}$ no longer occurs and one can use a fully anticommuting γ_5 . (For this reason the relations $H_{\mu\nu}^{AA} = H_{\mu\nu}^{VV}$ and therefore $H_{\mu\nu}^1 = H_{\mu\nu}^{P.C.} = H_{\mu\nu}^{VV}$ and $H_{\mu\nu}^4 = 0$ are still valid).

In other cases one has to ask how to escape from the

contradiction (A10). Since $\text{Tr} \gamma_5 \gamma_{\alpha} \gamma_{\beta} \gamma_{\gamma} \gamma_{\delta} = 0$ would make all parity odd calculations to vanish, three essential ways are open:

1. maintain (A8) and accept thereby (A10)
2. change condition (A9)
3. give up the fully anticommuting γ_5 , i.e. change condition (A8).

We briefly discuss the three methods and will see that only the last one is successful.

Chanowitz's proposal

The authors of /30,31/ decided to work with an anticommuting γ_5 (A8) and to accept (A10). Then all other traces get ill-defined and ambiguous. In order to give a meaning to the traces they introduced free parameters. As a consequence, contractions are no longer allowed, in particular $\gamma_1 \gamma_1 \neq 1^2$ (1 is a n-dimensional integration momentum²). As an example consider the trace $\text{Tr} \gamma_5 \gamma_{\mu} \gamma^{\mu} \gamma_{\alpha} \gamma^{\alpha} \gamma_{\beta} \gamma^{\beta} \gamma_{\gamma} \gamma^{\gamma}$. The authors of /30,31/ assigned a one-parameter (a_1)-ambiguity to its value according to the two possible ways in which it can be evaluated

$$(A11) \quad \text{Tr} \gamma_5 \gamma_{\mu} \gamma^{\mu} \gamma_{\alpha} \gamma^{\alpha} \gamma_{\beta} \gamma^{\beta} \gamma_{\gamma} \gamma^{\gamma} = \left[a_1 n + (1-a_1)(8-n) \right] 4i\epsilon(\alpha\beta\gamma\delta)$$

In their calculation of the VVA triangle anomaly the free parameter was fixed by postulating the "correct" value for the anomaly.

It is not clear whether the scheme of /30,31/ can be consistently formulated in more general situations. In particular, it is not clear whether the prescription of /30,31/ is applicable to higher-order IR/M calculations with their multiple γ -contractions inside parity odd traces. These multiple γ -contractions would necessitate the introduction of multifold parameter ambiguities and it is not clear whether one could find

²This would eliminate a free parameter, like e.g. the one in (A11).

Symbolically we can write

(A20) $4 = n + (4-n)$

The 4- dimensional space is decomposed into a direct sum of n- and (4-n)- dimensional subspaces corresponding to

(A21) $\tilde{g} \cdot g = g, \quad \tilde{g} \cdot \tilde{g} = \tilde{g}, \quad g \cdot g = 0$

We note that in particular for any (especially for a loop-) momentum p we have $\tilde{p}_\mu \equiv 0$.

However, as loop momenta are n-dimensional, in actual traces over a γ_5 and (4-dimensional) Dirac matrices γ_μ , only their "n-dimensional" parts occur³ ($\int d^n l \, l^\alpha \gamma^\beta \rightarrow g_{\alpha\beta} \gamma^\beta$). Then with the n-dimensional γ -matrices (compare (A14))

(A22) $\overline{\gamma}_\mu = g_{\mu\nu} \gamma^\nu$

equations (A8) and (A19) give

(A23) $\left\{ \overline{\gamma}_\mu, \gamma_5 \right\} = 0 \quad \text{with} \quad \overline{\gamma}_\mu \overline{\gamma}^\mu = n$

which rules out this possibility as in the Chanowitz's case.

A similar inconsistency follows from the characteristic

relation (A19) /13,80/ when using totally antisymmetric objects

see App. B:

(A24) $n(n-1)(n-2)(n-3)(n-4) = 0$

Thus dimensional reduction is ruled out for theories

containing a γ_5 or $\epsilon(\alpha,\beta,\gamma,\delta)$.

One might think that dimensional reduction leads to the same physical results as dimensional regularization if there are no parity- odd fermion loops /23/. Not so well known is the fact that even without γ_5 or $\epsilon(\alpha,\beta,\gamma,\delta)$ dimensional reduction can lead

(A19) $g_\mu^\rho \tilde{g}_{\rho\nu} = g_{\mu\nu} \quad ("n \times 4 = n")$

Relation (A19) is characteristic of dimensional reduction.

³Of course, the appearance of both "n-" and "4-dimensional" Dirac matrices complicates a calculation.

enough physical conditions to fix them. Also, since contractions are forbidden, one has: $\gamma_1 \cdot \gamma_1 \neq 1^2$ which spoils the validity of higher order field equations.

Dimensional Reduction

Siegel in his dimensional reduction scheme /99/ (see also /9,13,38/) attempts to keep the Dirac algebra in 4 dimensions and vector fields also, being split into a "n- dimensional" one plus (4-n) scalars; whereas the integrations are in n dimensions. A "4-dimensional" metric tensor $\tilde{g}_{\mu\nu}$ is then introduced and split into the "n-dimensional" one plus a (4-n)- dimensional one:

(A12) $\tilde{g}_{\mu\nu} = g_{\mu\nu} + \tilde{g}_{\mu\nu}$

which satiesfy

(A13) $\tilde{g}_\mu^\mu = 4, \quad g_\mu^\mu = n, \quad \tilde{g}_\mu^\mu = 4-n$

Dimensional reduction is then arrived at by postulating

(A14) $\gamma_\mu \equiv \tilde{g}_{\mu\nu} \gamma^\nu$

and

(A15) $p_\mu \equiv g_{\mu\nu} p^\nu$

Thus, γ_5 remains anticommuting (A8), but (A9) is changed to

(A16) $\gamma_\mu \gamma^\mu = 4$

For the validity of field equations and in order to preserve gauge invariance one needs the identity /23/

(A17) $\gamma p \gamma p \equiv p^2$

or

(A18) $p^\mu p^\nu (\tilde{g}_{\mu\nu} - g_{\mu\nu}) \equiv 0$

for any p, which leads to:

to ambiguous results /13/. It is the traces of 10 or more γ -matrices that are the origin of computational ambiguities, see App. B. We can thus expect "right" results up to 2-loops. Such ambiguities can be avoided if one avoids to assign a definite value to determinants of "4-dimensional" objects, i.e. if one considers them as parameters⁴ /13/. It seems (at least for non-supersymmetric theories) that the effort required for the use of dimensional reduction is too high, in particular when one bears in mind that there is a consistent, practicable version of dimensional regularization as we will see below.

The above analysis can be repeated for supersymmetric theories in D dimensions (D is the integer positive valued space time dimension) when replacing $4 \rightarrow D$ and $(4-n)$ by $(D-n)$ /13/. Then we can conclude that dimensional reduction should work at least up to traces with no more than $(2D+2)$ γ -matrices.

The BM scheme

A systematic and consistent formulation of dimensional

⁴The implementation of such a prescription on an algebraic computer program like REDUCE is problematic. In particular, since certain simplifications of scalar products of γ -matrices /34/ and algorithms like the Kahane algorithm /60/ are no longer allowed, the calculations will spend much more time.

regularization⁵ is the original one of 't Hooft and Veltman /56/ and Akyeampong and Delbourgo /5/ afterwards systematized by Breitenlohner and Maison /25/ in the following referred to as the BM γ_5 -scheme.

One starts from the observation that n - and 4 -dimensional covariants are required due to the genuine 4 -dimensionality of the ϵ -tensor. The n -dimensional space is decomposed into a 4 -dimensional one plus a $(n-4)$ -dimensional one. Symbolically one writes⁶

$$(A25) \quad n = 4 + (n-4)$$

This is realized by correspondingly splitting the n -dimensional metric tensor $g_{\mu\nu}$

$$(A26) \quad g_{\mu\nu} = \tilde{g}_{\mu\nu} + \tilde{g}_{\mu}^{\mu} \tilde{g}_{\nu}^{\nu} \quad , \quad \tilde{g}_{\mu}^{\mu} = n \quad , \quad \tilde{g}_{\nu}^{\nu} = 4 \quad , \quad \tilde{g}_{\mu}^{\mu} = n-4$$

The "orthogonality (A25)" then reads

$$(A27) \quad g_{\mu}^{\rho} \tilde{g}_{\rho\nu} = \tilde{g}_{\mu\nu} \quad , \quad g_{\mu}^{\rho} \tilde{g}_{\rho\nu} = \tilde{g}_{\mu\nu} \quad , \quad \tilde{g}_{\mu}^{\rho} \tilde{g}_{\rho\nu} = 0$$

Corresponding covariants as p_{μ} , $\tilde{p}_{\mu}$, $\tilde{\gamma}_{\mu}$, $\tilde{\gamma}_{\mu}$ are introduced according to: $\tilde{\gamma}_{\mu} \equiv \tilde{g}_{\mu}^{\nu} \gamma_{\nu}$ etc. .

⁵As far as we know this is the only consistent formulation of dimensional regularization. The question is open whether the γ_5 -scheme of /103/ provides a consistent formulation of the γ_5 -problem. Work is in progress whether the non-uniqueness of their parameter b leads to a unique result /4/.

⁶Compare (A25) with the analogous decomposition of dimensional reduction (A20).

The 4-dimensionality of the ϵ -tensor is kept by defining

$$(A28) \quad \epsilon(\mu_1 \mu_2 \mu_3 \mu_4) \epsilon(\nu_1 \nu_2 \nu_3 \nu_4) = - \det(\tilde{g}_{\alpha\beta}) \quad \begin{matrix} \alpha = \mu_1 \\ \beta = \nu_j \end{matrix}$$

Note that the 4-dimensional metric tensor appears on the RHS of

(A28). As a consequence we have

$$(A29) \quad \tilde{g}_{\mu\nu} \epsilon(\nu\rho\kappa\lambda) = 0$$

i.e. the ϵ -tensor acts in 4 dimensions only. Analogously, the

Schouten identity is valid for 4-dimensional covariants only:

$$(A30) \quad \tilde{g}_{\alpha\mu_1} \epsilon(\mu_2 \mu_3 \mu_4 \mu_5) + \text{cycl.}(\mu_1) = 0$$

Defining γ_5 by

$$(A31) \quad \gamma_5 = \frac{i}{4!} \epsilon(\mu_1 \mu_2 \mu_3 \mu_4) \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4}$$

one can prove $\gamma_5^2 = 1$ and its commutation relation /25/

$$(A32) \quad \{\gamma_\alpha, \gamma_5\} = 2 \gamma_\alpha \gamma_5$$

We find that only the "4-dimensional" γ -matrices anticommute with

γ_5 while the (n-4)-dimensional ones commute:

$$(A33) \quad \left\{ \tilde{\gamma}_\alpha, \gamma_5 \right\} = 0, \quad \left[\tilde{\gamma}_\alpha, \gamma_5 \right] = 0$$

Traces involving γ_5 are calculated by substituting γ_5 by its definition (A31) and eliminating ϵ -tensors by (A28). The above prescriptions lead to a unique normal form for all traces.

That the BM scheme provides a consistent treatment of the singularities in the UV realm (correct anomalies, all order renormalizability) has been demonstrated in /25,26,104/.

That the BM scheme also provides a consistent and practical calculational scheme for the treatment of IR/M singularities will be demonstrated in the main text (§5) where the p.v. hadron tensors are calculated.

Appendix B: The inconsistency inherent in dimensional reduction

In this section we demonstrate the fact that dimensional reduction (DR) can lead to ambiguous results even when no γ_5 or ϵ -tensor is present. To this end we first exhibit the inconsistency inherent in dimensional reduction.

In the following we follow closely the analysis of /13/.

Since we are interested in perturbative QCD calculations we consider the DR in 4 dimensions. The generalization to D dimensions appropriate for supersymmetric theories in D dimensions will not be treated.

Dimensional reduction is by construction manifestly supersymmetric but inconsistent. The reason is that the decomposition defining dimensional reduction

$$(B1) \quad 4 = n + (4-n)$$

is itself contradictory for non-integer¹ n. It conflicts with the familiar property of the 4-dimensional ϵ -tensor. We define:

$$(B2) \quad \det(\mu_1 \dots \mu_k \nu_1 \dots \nu_k) = \begin{vmatrix} g_{\mu_1 \nu_1} & \dots & g_{\mu_k \nu_1} \\ \vdots & \ddots & \vdots \\ g_{\mu_1 \nu_k} & \dots & g_{\mu_k \nu_k} \end{vmatrix}$$

for any non-negative integer k and analogous $\det()$ replacing $g_{\mu\nu}$ in (B2) by $\tilde{g}_{\mu\nu}$. (Here $\tilde{g}_{\mu\nu}$ is the metric tensor in the n-(4-) dimensional space, see (A13)).

Now we make use of the identity

¹ Writing the decomposition (B1) one assumes n to be: $n = 4 - \epsilon$, where ϵ is small.

$$(B3) \quad g_{\mu_1 \nu_1} \dots g_{\mu_k \nu_k} \det(\mu_1 \dots \mu_k, \nu_1 \dots \nu_k) = n(n-1) \dots (n-k+1)$$

(B3) implies that for an arbitrary integer k the determinant (B2) is not identically zero (n was assumed to be non-integer).

Consequently, the Lorentz indices in the n -dimensional space² may be viewed as running through infinite number of distinct values.

However, this is precisely a definition of an infinite-dimensional space. The origin of the inconsistencies inherent in

(B1) now becomes clear: decomposing the 4-dimensional space according to (B1) is really a decomposing of a 4-dimensional space into a direct sum of two infinite-dimensional spaces.

Apparently, this is impossible. We will see in (B7) the impossibility explicitly.

The characteristic relation of dimensional reduction

$$(B4) \quad \tilde{g}_{\mu\rho} g^\rho_\nu = g_{\mu\nu}$$

allows us to write $\tilde{\det}$ instead of $\det$ in (B3):

$$(B5) \quad g_{\mu_1 \nu_1} \dots g_{\mu_k \nu_k} \tilde{\det}(\mu_1 \dots \mu_k, \nu_1 \dots \nu_k) = n(n-1) \dots (n-k+1)$$

But for $k > 4$

$$(B6) \quad \tilde{\det}(\mu_1 \dots \mu_k, \nu_1 \dots \nu_k) \equiv 0$$

since a tensor which is totally antisymmetric in $k > 4$ indices can not exist in 4 dimensions. We arrive at the contradiction /13,80/:

$$(B7) \quad 0 = g_{\mu_1 \nu_1} \dots g_{\mu_5 \nu_5} \tilde{\det}(\mu_1 \dots \mu_5, \nu_1 \dots \nu_5) = n(n-1) \dots (n-4)$$

²The determinant (B2) is defined by use of the n -dimensional metric tensor $g_{\mu\nu}$, it is thus an object "living" in the n -dimensional subspace (see (B1)) of the 4-dimensional space.

Eq. (B7) implies that it is possible for DR to produce

ambiguous results. This can occur even when calculating Feynman diagrams not containing a γ_5 . Consider a trace of the product of

$2k$ γ -matrices. Remembering that γ -matrices are 4-dimensional

$$(B8) \quad \left\{ \gamma_\mu, \gamma_\nu \right\} = 2\tilde{g}_{\mu\nu} \quad 1$$

we obtain the well known result

$$(B9) \quad \text{Tr} \gamma_{\mu_1} \dots \gamma_{\mu_{2k}} = \text{Tr} 1$$

$$\left[\tilde{g}_{\mu_1 \mu_2} \tilde{g}_{\mu_3 \mu_4} \dots \tilde{g}_{\mu_{2k-1} \mu_{2k}} - \tilde{g}_{\mu_1 \mu_3} \tilde{g}_{\mu_2 \mu_4} \dots \tilde{g}_{\mu_{2k-1} \mu_{2k}} + \dots \right]$$

with $1 \cdot 3 \cdot 5 \dots (2k-1)$ independent terms inside the square

brackets. However, for $k > 4$ there exists the Kahane algorithm

/60/ for reducing the total number of terms on the RHS of (B9).

An effect of the algorithm is essentially the following: form the combinations proportional to $\tilde{\det}(\mu_1 \dots \mu_5, \nu_1 \dots \nu_5)$ and then delete them according to (B6).

Now the Feynman diagram realization of Siegel's contradiction

(B7) becomes transparent: explicit expressions for the trace (B9)

derived with and without use of the Kahane algorithm can differ

from each other by a quantity which is proportional to this

particular determinant. This difference survives in the final

result, if subsequent contractions of Lorentz indices do not

eliminate the determinant, e.g. if the contractions are of the

form (B5).

Thus, it is a trace of $(2 \cdot 4 + 2)$ or more γ -matrices that appears to be an origin of computational ambiguities in DR. For higher order calculations one could attempt to specify the

"dangerous" terms and hence determine the domain of consistency of DR. Another way is to abandon the first equality (B7) and consider that the determinants $\tilde{\det}(\mu_1 \dots \mu_k, \nu_1 \dots \nu_k)$ no longer have a value; they must be considered as parameters /13/.

parts in the integrated 4-parton cross sections do indeed vanish. The complete angular integration of $\tilde{p}_4^2$ can of course be

obtained more directly. Making an ansatz for the LHS of (C8) and remembering that $\int d\hat{n} = 1$ (see (4.2.24)) we easily find

$$(C8) \quad \int d\hat{n}_{34} p_{4\alpha} p_{4\beta} = \frac{1}{4(1-n)} \left[s_{34} g_{\alpha\beta} - n(p_3 + p_4)_\alpha (p_3 + p_4)_\beta \right]$$

Then noting that $\tilde{p}_4^2 \equiv \tilde{g}^{\alpha\beta} p_{4\alpha} p_{4\beta}$ we obtain

$$(C9) \quad \int d\hat{n}_{34} \tilde{p}_4^2 = \frac{-\epsilon}{2(3-2\epsilon)} s_{34}$$

Appendix C: n-dimensional integral of the 4-dimensional scalar $\tilde{p}_4^2$

Consider the explicit representation of the n-dimensional momenta in the (3-4) cms frame

$$(C1) \quad p_3 = \frac{1}{2} \sqrt{s_{34}} (1, \pm \sin \theta_1 \dots \sin \theta_{n-3}, \dots, \pm \sin \theta_1 \cos \theta_2, \pm \sin \theta_1 \cos \theta_1, \pm \cos \theta_1)$$

$$(C2) \quad p_2 = \frac{s_{234} - s_{34}}{2\sqrt{s_{34}}} (1, 0, \dots, 0, 1)$$

$$(C3) \quad p_1 = \frac{s_{134} - s_{34}}{2\sqrt{s_{34}}} (1, 0, \dots, 0, \sin \theta, \cos \theta)$$

From (C1) we get

$$(C4) \quad \begin{aligned} \tilde{p}_4^2 &\equiv p_{40}^2 - p_{4x}^2 - p_{4y}^2 - p_{4z}^2 \\ &= \frac{s_{34}}{4} (1 - \sin^2 \theta_1 \cos^2 \theta_2 - \sin^2 \theta \cos^2 \theta_1 - \cos^2 \theta) \\ &= \frac{s_{34}}{4} \sin^2 \theta_1 \sin^2 \theta_2 \\ &= s_{34} v(1-v) \sin^2 \theta_1 \sin^2 \theta_2 \end{aligned}$$

where we have set $v \equiv \frac{1}{2} (1 - \cos \theta)$.

Compared to the case where n-dimensional scalars have to be integrated, $\tilde{p}_4^2$ introduces an additional θ_2 -dependence in the integrand. This can be taken into account by the replacement

$$(C5) \quad 1 \rightarrow \frac{1}{N_{\theta_2}} \int_0^\pi d\theta_2 \sin^{-1-2\epsilon} \theta_2$$

in the integrand where

$$(C6) \quad N_{\theta_2} = \frac{r(-\epsilon) \sqrt{\pi}}{r(\frac{1}{2} (1-2\epsilon))}$$

The θ_2 -integration of $\tilde{p}_4^2$ gives

$$(C7) \quad \frac{1}{N_{\theta_2}} \int_0^\pi d\theta_2 \sin^{-1-2\epsilon} \theta_2 \tilde{p}_4^2 = s_{34} v(1-v) \sin^2 \theta_1 \frac{-2\epsilon}{1-2\epsilon}$$

which can be used in (5.3.7,9) to show that the nonfactorized IR/M

Appendix D: Renormalization details

In this section we will deduce the following rule for calculating a scattering amplitude to order g^3 ($\alpha_s \equiv g^2/4n$) in the $\overline{MS}$ renormalization scheme:

"Take all possible topologically distinct diagrams and weight those with self-energy insertions at the external legs by a factor $1/2$. Then remove the UV poles by just dropping them."

A remark concerning the $\overline{MS}$ scheme will be given at the end of this section.

We will motivate the validity of the rule stated above in the example of Thomson scattering in QED which defines the physical electric charge e_{ph} . The derivation of the corresponding amplitude will be done to any order of perturbation theory. The generalization to processes with an arbitrary number of external legs and/or to QCD processes should then be obvious. (In particular, the rules concerning the external legs do not depend on the species of the associated particles: it is all the same whether they are gluons or photons, quarks or electrons. The QED example is chosen only to avoid unnecessary lengthy expressions).

To start, we briefly recapitulate some well known facts about renormalization /92/. We do not want to give a systematic review of renormalization in general, but rather collect some formulae which will clarify the way how to calculate and renormalize specific UV divergent amplitudes. We restrict ourselves to the case of massless QCD (putting $N_C = 0$, $C_F = 1$ and $N_f = 2$ we get the corresponding QED results) and use the Feynman gauge.

In the following we denote by "B" the bare or unrenormalized

quantities and by "R" the renormalized ones. We assume that the (UV divergent) bare quantities are dimensionally regularized. The question how to disentangle the UV and simultaneously identify IR/M poles is dealt with in the main text. Since we perform our calculation in the minimal subtraction (MS) scheme we will mainly consider its renormalization conditions. However, for readers more familiar with the on shell (on) scheme we also add the corresponding on shell properties. (Clearly the use of the on shell scheme in a massless theory is a problematic one; although the on shell renormalized Greens functions are IR/M divergent, all divergences can be removed since the strong coupling constant g is IR/M divergent too. For an example see /98/).

First we discuss the renormalization constants Z_1 , Z_2 and Z_3 .

The renormalization constant Z_3

of the gluon wave function. Denoting the vacuum polarization by (q is the gluon momentum)

$$(D1) \quad \Pi_{\mu\nu}(q) = i(-q^2 g_{\mu\nu} + q_\mu q_\nu) \Pi(q^2)$$

the bare gluon propagator $G_{\mu\nu}^B$ is given by (up to terms proportional to $q_\mu q_\nu$):

$$\begin{aligned} (D2) \quad G_{\mu\nu}^B(q) &= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2} (-i)q_\rho^2 g_{\rho\sigma} \Pi(q^2) \frac{-ig_{\sigma\nu}}{q^2} + \dots \\ &= \frac{-ig_{\mu\nu}}{q^2} \left[1 - \Pi(q^2) + \dots \right] \\ &= \frac{-ig_{\mu\nu}}{q^2} \left[1 + \Pi^B(q^2) \right] \end{aligned}$$

The constant Z_3 is introduced in order to remove the UV divergences

$$(D3) \quad 1 + \Pi(q^2) = Z_3 \left[1 + \Pi^B(q^2) \right] = UV \text{ finite}$$

The renormalized boson propagator is then

$$(D4) \quad G_{\mu\nu}^R(q) = Z_3^{-1} G_{\mu\nu}^B(q)$$

The (UV-) finite terms of Z_3 depend on the renormalization scheme. The renormalization condition for Z_3 in the two considered schemes is

$$(D5) \quad \text{on shell scheme (on):} \quad \pi^{\text{on}}(0) \equiv 0$$

$$(D6) \quad \text{MS scheme (MS):} \quad \pi^{\text{MS}}(q^2) \equiv \pi^{\text{B}}(q^2)_{\text{FP}}$$

where FP denotes the UV finite part. This gives for the renormalization constant

$$(D7) \quad Z_3^{\text{on}} = \frac{1}{1 + \pi^{\text{B}}(0)}, \quad Z_3^{\text{MS}} = \frac{1}{(1 + \pi^{\text{B}})_{\text{UP}}}$$

where UP denotes the UV pole (We split $\pi^{\text{B}} \equiv \pi_{\text{FP}}^{\text{B}} + \pi_{\text{UP}}^{\text{B}}$). Then we get for the renormalized propagators in the respective schemes:

$$(D8) \quad G_{\mu\nu}^{\text{on}}(q) = \frac{-ig_{\mu\nu}}{q^2(1 + \pi^{\text{B}}(q^2) - \pi^{\text{B}}(0))}$$

$$G_{\mu\nu}^{\text{MS}}(q) = \frac{-ig_{\mu\nu}}{q^2(1 + \pi^{\text{B}}(q^2) - \pi_{\text{UP}}^{\text{B}})}$$

Since the bare propagator behaves like

$$(D9) \quad G_{\mu\nu}^{\text{B}} \rightarrow -ig_{\mu\nu}/(q^2) \cdot Z_3/(1 + \pi_{\text{R}}(0))$$

when q^2 approaches the mass shell, we have to associate to the external boson line with the polarization vector ϵ_μ a factor

$$(D10) \quad \left[\frac{1 + \pi^{\text{R}}(0)}{Z_3} \right]^{1/2} \epsilon_\mu$$

Note that this factor reduces to $Z_3^{-1/2}$ in the on shell scheme¹.

Writing the renormalization constants as

$$(D11) \quad Z = 1 + \delta Z$$

¹An explanation of the square root appearing in (D10) can be found e.g. in [24], nevertheless we will add some comments concerning this factor at the end of this section.

we get to $O(\alpha_s)$: $Z^{-1} = 1 - \delta Z$, $\sqrt{Z} = 1 + \frac{1}{2} \delta Z$, etc. . Using that π^{B} is itself of order α_s we have $(n = 4 - 2\epsilon)^2$ (see e.g. [98])

$$(D12) \quad \delta Z_3^{\text{on}} = -\pi^{\text{B}}(0) = -\frac{\alpha_s^{\text{on}}}{2\pi} \left\{ -\frac{5}{6} N_C + \frac{1}{3} N_f \right\} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right)$$

$$(D13) \quad \delta Z_3^{\text{MS}} = -(\pi^{\text{B}})_{\text{UP}} = -\frac{\alpha_s^{\text{MS}}}{2\pi} \left\{ -\frac{5}{6} N_C + \frac{1}{3} N_f \right\} \frac{1}{\epsilon_{\text{UV}}}$$

where ϵ_{UV} (ϵ_{IR}) denotes the UV (IR) pole).

The renormalization constant Z_2

of the fermion wave function. We denote by $(-i)\Sigma(p)$ the fermion self energy (p is the fermion momentum). Since the fermions are assumed to be massless we have

$$(D14) \quad -i\Sigma(p) = -i \gamma p \Sigma_2(p^2)$$

The bare fermion propagator is given by

$$(D15) \quad S^{\text{B}}(p) = \frac{1}{\gamma p} + \frac{i}{\gamma p} (-i\Sigma^{\text{B}}(p)) \frac{1}{\gamma p} + \dots \\ = \frac{1}{\gamma p} (1 + \Sigma_2^{\text{B}}(p^2) + \dots) \\ = \frac{1}{\gamma p - \Sigma^{\text{B}}(p)} = \frac{1}{\gamma p - \Sigma_2^{\text{B}}(p^2)} \frac{1}{\gamma p}$$

The constant Z_2 makes the self energy finite. It is defined by

$$(D16) \quad \gamma p - \Sigma^{\text{R}}(p) = Z_2 [\gamma p - \Sigma^{\text{B}}(p)] = \text{UV finite}$$

Using (D14) we obtain

$$(D17) \quad 1 - \Sigma_2^{\text{R}}(p^2) = Z_2 \left[1 - \Sigma_2^{\text{B}}(p^2) \right]$$

The renormalized fermion propagator is defined by

$$(D18) \quad S^{\text{R}}(p) = Z_2^{-1} S^{\text{B}}(p)$$

²When renormalizing a Greens function: $r^{\text{R}} = Z_r r^{\text{B}}$ one has to distinguish carefully the renormalized coupling g_{R} and the bare coupling g_{B} . r^{R} and Z_r contain g_{R} , whereas r^{B} depends on g_{B} . However, at the one-loop level, we may replace g_{B} by g_{R} (or equivalently g_{S}^{B} by g_{S}^{R}) or vice versa in Z_r or r^{B} .

In the two schemes considered we have

$$(D19) \quad z_2^{\text{on}}(p^2=0) = 0, \quad z_2^{\text{MS}}(p^2) = z_2^{\text{B}}(p^2)_{\text{FP}}$$

This defines the constant Z_2 to be

$$(D20) \quad z_2^{\text{on}} = \frac{1}{1 - z_2^{\text{B}}(0)}, \quad z_2^{\text{MS}} = \frac{1}{(1 - z_2^{\text{B}})_{\text{UP}}}$$

The renormalized fermion propagator can then be written as:

$$(D21) \quad S^{\text{on}}(p) = \frac{1}{1 - z_2^{\text{B}}(p^2) - z_2^{\text{B}}(0)} \frac{i}{\gamma p}$$

$$(D22) \quad S^{\text{MS}}(p) = \frac{1}{1 - z_2^{\text{B}}(p^2) - (z_2^{\text{B}})_{\text{UP}}} \frac{i}{\gamma p}$$

Unitarity requires that $\gamma p - m_R - z^R(p)$ has a single pole at the physical mass m_{ph} . This is usually written as

$$(D23) \quad \gamma p - m_R - z^R(p) \rightarrow (1 + f_2)(\gamma p - m_{\text{ph}}) \quad \text{as } \gamma p \rightarrow m_{\text{ph}}$$

In the massless case ($m_{\text{ph}}=0$) we find

$$(D24) \quad f_2 = -z_2^R(0)$$

Since the bare propagator

$$(D25) \quad S^{\text{B}}(p) \rightarrow \frac{i}{\gamma p} z_2/(1+f_2)$$

as $\gamma p \rightarrow 0$ we have to associate to every external line with the

spinor $u(\bar{u}, v, \bar{v})$ a factor

$$(D26) \quad \left[\frac{1 + f_2}{z_2} \right]^{1/2} u$$

Note that this factor reduces to $z_2^{-1/2}$ in the on shell scheme.

To order α_s we find

$$(D27) \quad \delta z_2^{\text{on}} = z_2^{\text{B}}(0) = -\frac{\alpha_s}{2\pi} C_F \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right)$$

$$(D28) \quad \delta z_2^{\text{MS}} = (z_2^{\text{B}})_{\text{UP}} = -\frac{\alpha_s^{\text{on}} C_F}{2\pi} \frac{1}{z} \frac{1}{\epsilon_{\text{UV}}}$$

The renormalization constant Z_1

of the fermion-gluon-fermion vertex. We denote by $ig_B \gamma_\mu^{\text{B}}(q, p', p)$ (p' and p are the fermion momenta and $q \equiv p' - p$ is the gluon

momentum) the complete one particle irreducible three point function. It is usually split into the lowest order term plus a remainder

$$(D29) \quad \gamma_\mu^{\text{B}}(q, p', p) = \gamma_\mu + r_\mu^{\text{B}}(q, p', p)$$

The constant Z_1 makes the vertex function finite

$$(D30) \quad \gamma_\mu + r_\mu^{\text{B}}(q, p', p) = Z_1 \left[\gamma_\mu + r_\mu^{\text{B}}(q, p', p) \right] = UV \text{ finite}$$

The renormalization condition for the three point function reads

$$(D31) \quad \bar{u}(p') \left\{ r_\mu^{\text{on}}(q, p', p) \right\}_{\substack{p^2=p'^2=0 \\ p=p'}} u(p) = 0$$

$$r_\mu^{\text{MS}}(q, p', p) = r_\mu^{\text{B}}(q, p', p)_{\text{FP}}$$

The Lorentz structure of r_μ for off shell (massless!) particles³ is

$$(D32) \quad r_\mu(q, p', p) = \gamma_\mu \gamma_1 + p_\mu \gamma p \gamma_2 + p_\mu \gamma p' \gamma_3 +$$

$$p'_\mu \gamma p \gamma_4 + p'_\mu \gamma p' \gamma_5 + \gamma p' \gamma_\mu \gamma p \gamma_6$$

where $r_1 = r_1(q^2, p^2, p'^2)$.

Of course, in a renormalizable field theory at the one loop level the only UV divergent form factor is r_1 . Thus we can write

(D30) to $O(\alpha_s)$ as

$$(D33) \quad 1 + r_1^{\text{R}} = Z_1(1 + r_1^{\text{B}})$$

In the two schemes under consideration we get

$$(D34) \quad z_1^{\text{MS}} \equiv 1 + \delta z_1^{\text{MS}} = \frac{1}{(1 + r_1^{\text{B}})_{\text{UP}}} = 1 - (r_1^{\text{B}})_{\text{UP}}$$

$$(D35) \quad z_1^{\text{on}} \equiv 1 + \delta z_1^{\text{on}} = \frac{1}{1 + r_1^{\text{B}}(0,0,0)} = 1 - r_1^{\text{B}}(0,0,0)$$

Explicitly we have

³Note that for on shell fermions between spinors $\bar{u}(p') \dots u(p)$ only r_1 survives.

the scattering amplitude becomes

$$(D41) \quad S_{\mu}(q) = \frac{Z_2}{1 + f_2} \frac{[Z_3(1 + \pi^R(0))]^{1/2}}{1 + \pi^R(q^2)} \bar{u}(p') i e_B \gamma_{\mu}^B(q) \epsilon^{\mu} u(p)$$

Of course, we could have started from this expression by using the Feynman rules: draw all possible topologically distinct diagrams but without diagrams containing self-energy insertions at the external legs, and associate with each spinor a factor $[Z_2/(1 + f_2)]^{1/2}$ and with each polarization vector a factor $[Z_3/(1 + \pi^R(0))]^{1/2}$.

Now we use the fact that $\gamma_{\mu}^B = Z_1^{-1} \gamma_{\mu}^R$ and the definition of the renormalized charge e_R

$$(D42) \quad e_B = \mu_R^{\epsilon} \frac{Z_1}{Z_2 Z_3^{1/2}} e_R$$

and arrive at

(D43)

$$S_{\mu}(q) = \frac{(1 + \pi^R(0))^{1/2}}{1 + \pi^R(q^2)} \frac{1}{1 + f_2} \bar{u}(p') i e_R^{\epsilon} \mu_R^{\epsilon} \gamma_{\mu}^R(q) \epsilon^{\mu} u(p)$$

The Thomson amplitude defining the physical electric charge via $i e_{ph}^{\epsilon} \gamma_{\mu}^{\epsilon}$ is finally arrived at in the limit $q \rightarrow 0$ in (D43).

Thus we end up with

$$(D44) \quad e_{ph}^{\epsilon} \gamma_{\mu}^{\epsilon} = e_R^{\epsilon} \left[\frac{\mu_R^{\epsilon}}{\mu_{ph}^{\epsilon}} \right] \left[\frac{1}{1 + \pi^R(0)} \right]^{1/2} \frac{1}{1 + f_2} \left[\gamma_{\mu}^R + r_{\mu}^R(0) \right]$$

Since the on shell scheme is defined by $r_{\mu}^{on}(0) = 0$, $f_2^{on} = 0$ and $\pi^{on}(0) = 0$ we find (setting $\mu_{on} \equiv \mu_{ph}$)

$$(D45) \quad e_{ph}^{\epsilon} \equiv e_{on}$$

For on shell spinors the 3-point function γ_{μ} clearly depends only on the photon momentum q and is (in the massless case) proportional to γ_{μ} .

$$(D36) \quad \sigma_{11}^{on} = - \frac{\alpha_s^{on}}{2\pi} \frac{N_C + C_F}{2} \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right)$$

$$(D37) \quad \sigma_{11}^{MS} = - \frac{\alpha_s^{MS}}{2\pi} \frac{N_C + C_F}{2} \frac{1}{\epsilon_{UV}}$$

We are now in position to describe how to calculate a scattering amplitude. To clarify the procedure we consider the Thomson scattering in QED /91/ which defines the physical electric charge e_{ph} .

The full bare $e^-e^+\gamma$ vertex is given by

$$(D38) \quad \gamma_{\mu}^B(p', p) = G_{\mu\nu}^B(q) S^B(p') i e_B \gamma_{\nu}^B(q, p', p) S^B(p)$$

For the scattering amplitude we need the truncated vertex, i.e. we have to amputate the external e^+ , e^- and γ legs by multiplying with the inverse complete physical propagators $(-i)\gamma p', (-i)\gamma p$ and $i q^2$, respectively. Recalling the factors

$$\left[\frac{(1 + f_2)/Z_2}{(1 + f_2)/Z_2} \right]^{1/2} \bar{u}(p'), \left[\frac{(1 + f_2)/Z_2}{(1 + f_2)/Z_2} \right]^{1/2} \bar{u}(p) \text{ and } \left[\frac{(1 + \pi_R(0))/Z_3}{(1 + \pi_R(0))/Z_3} \right]^{1/2} \epsilon^{\mu}$$

for the antifermion, fermion spinor and photon polarization

vector, respectively, the scattering amplitude is

$$(D39) \quad S_{\mu}(q, p', p) = \frac{1 + f_2}{Z_2} \left[\frac{1 + \pi_R(0)}{Z_3} \right]^{1/2} \bar{u}(p') (-i)\gamma p' S^B(p')$$

$$i e_B \gamma_{\nu}^B(q, p', p) G_{\mu\nu}^B(q) i q^2 \epsilon^{\mu} S^B(p) (-i)\gamma p u(p)$$

Setting the fermions on the mass shell we have (D25)

$$(D40) \quad (-i)\gamma p' S^B(p') \rightarrow \frac{Z_2}{1 + f_2}, \quad (-i)\gamma p S^B(p) \rightarrow \frac{Z_2}{1 + f_2}$$

$$\text{and using} \quad G_{\mu\nu}^B(q) i q^2 = \frac{Z_3 g_{\mu\nu}}{1 + \pi^R(q^2)}$$

In the MS scheme we get

$$(D46) \quad e_{ph} = (\mu_{MS}/\mu_{ph})^\epsilon e_{MS} \left[\frac{1}{1 + \pi^{MS}(0)} \right]^{1/2} \frac{1}{1 - \pi^{MS}(0)} \left[1 + r_1^{MS}(0) \right]$$

As a consistency check we can calculate the MS scattering amplitude from the on shell one by substituting e_{on} by e_{MS} /92,98/. To this end we use (D42) to write

$$(D47) \quad e_{on} = e_{MS} (\mu_{MS}/\mu_{on})^\epsilon \frac{Z_1^{MS} Z_2^{on}}{Z_1^{on} Z_2^{MS}} \frac{Z_3^{on}}{Z_3^{MS}} \left[\frac{Z_3^{on}}{Z_3^{MS}} \right]^{1/2}$$

Now we make use of

$$(D48) \quad \frac{1 + \pi^{on}(q^2)}{Z_3^{on}} = 1 + \pi^B(q^2) = \frac{1 + \pi^{MS}(q^2)}{Z_3^{MS}}$$

and the analogous relations for Z_3 and Z_1 and find⁵

$$(D49) \quad e_{on} = \frac{1 + r_1^{MS}(q^2, p'^2, p^2)}{1 + r_1^{on}(q^2, p'^2, p^2)} \frac{1 - \pi^{on}(p^2)}{1 - \pi^{MS}(p^2)} \left[\frac{1 + \pi^{on}(q^2)}{1 + \pi^{MS}(q^2)} \right]^{1/2} \\ = e_{MS} (\mu_{MS}/\mu_{on})^\epsilon \frac{1 + r_1^{MS}(q^2)}{1 + r_1^{on}(q^2)} \frac{1}{1 - \pi^{MS}(0)} \left[\frac{1}{1 + \pi^{MS}(0)} \right]^{1/2}$$

Finally we get for the amplitude for scattering a virtual

photon (momentum q) by an (on shell, $p'^2 = p^2 = 0$) fermion

$$(D50) \quad S_\mu(q^2) = ie_{on}\mu_{on}^\epsilon \left[1 + r_1^{on}(q^2) \right] \frac{1}{1 + \pi^{on}(q^2)} \bar{u}(p') \gamma_\mu \epsilon^\mu(q) u(p) \\ = ie_{MS}\mu_{MS}^\epsilon \bar{u}(p') \gamma_\mu \epsilon^\mu(q) u(p) \\ \left[1 + r_1^{MS}(q^2) \right] \frac{1}{1 - \pi^{MS}(0)} \left[1 + \pi^{MS}(0) \right]^{-1/2} \frac{1}{1 + \pi^{MS}(q^2)}$$

⁵ Note that (D49) is independent of the momenta since the renormalized quantities differ only by constants.

In the limit $q, 0$ we recover the result

$$(D51) \quad S_\mu(0) = ie_{on}\mu_{ph}^\epsilon \bar{u}(p') \gamma_\mu u(p) \\ = ie_{MS}\mu_{MS}^\epsilon \left[\frac{1}{1 + \pi^{MS}(0)} \right]^{1/2} \frac{1}{1 - \pi^{MS}(0)} \left[1 + r_1^{MS}(0) \right] \bar{u}(p') \gamma_\mu u(p)$$

We remark that we did not use the identity $Z_1 = Z_2$ (implying $r_1^B(0) = -\pi_2^B(0)$ and $(r_1^B)^{up} = -(\pi_2^B)^{up}$) valid in QED in order to illustrate the general procedure.

Relation (D47) can be used within perturbation theory to determine a counterterm transforming a result obtained in the on shell scheme to the result in the MS scheme /92,98/. However, we will show how the perturbative calculation can be done totally within the MS scheme. To this end we consider the above scattering amplitude with all particles on shell, i.e. $q^2 = p'^2 = p^2 = 0$. (This is the situation in the $O(g^3)$ calculation of the amplitude $\gamma, Z^0 \rightarrow q\bar{q}g$, where the quark, antiquark and gluon are taken on shell.)

In the MS scheme the on shell amplitude is given by (see (D43) for $q = 0$, (D25) and use $r_\mu = r_1 \gamma_\mu$) (D52)

$$S_\mu = ie_{MS}\mu_{MS}^\epsilon \left[1 + r_1^{MS}(0) \right] \left[1 + \pi^{MS}(0) \right]^{-1/2} \left[1 - \pi_2^{MS}(0) \right]^{-1} \bar{u}\gamma_\mu u$$

To order e^3 this can be written as

$$(D53) \quad S_\mu = ie_{MS}\mu_{MS}^\epsilon \left[1 + r_1^{MS}(0) + \pi_2^{MS}(0) - \frac{1}{2}\pi^{MS}(0) \right] \bar{u}\gamma_\mu u$$

The corresponding Feynman diagrams are shown in Fig. 39. We observe, that diagrams with self-energy insertions at the external legs obtain a factor 1/2. Since the MS quantities are just the bare ones without the UV pole (to $O(e^3)$!, see footnote 2), we arrive at the following rule for calculating an arbitrary amplitude in order e^3 (or g^3):

To this end we consider the bare fermion propagator

$$(D57) \quad (-i) \gamma^B S^B(p) = \frac{1}{1 - \Sigma_2^B(p^2)} = \frac{Z_2}{1 - \Sigma_2^B(p^2)}$$

We now imagine a physical process containing an external fermion with momentum p . After squaring the corresponding amplitude this line yields the full propagator (D57) with p on shell, i.e. $p^2=0$. Thus the amplitude must have had a factor given by the square root of (D57). But this is exactly the factor (D54). Therefore the external fermion line receives the factor (D54) where we identify $f_2 \equiv -\Sigma_2^R(0)$. Similarly, we arrive at the factor (D55) for an external boson.

Alternatively we can think of an external line as being a propagator where one vertex approaches infinity. After squaring we get back the propagator. Another argument confirming the square root appearing in (D54, D55) is given by the Ward identity in QED. We use it in the form

$$(D58) \quad \Gamma_\mu(p, p) = - \frac{\partial}{\partial p^\mu} \Sigma(p)$$

Translated into the Feynman diagrams for e.g. the process $\gamma \rightarrow e^+ e^-$ this means that the UV poles of the vertex correction cancel already against one diagram containing a fermion self energy correction. Or if one includes two diagrams, with self energy corrections for the fermion and the antifermion, respectively, then these two diagrams have to be weighted by a factor $1/2$ compared to the vertex correction diagram. Finally we note that the above mentioned arguments are not restricted to one loop.

Take all possible topologically distinct diagrams and weight those with self-energy insertions at the external legs by a factor $1/2$. Then remove the UV poles by just dropping them.

Sometimes the result is desired in the $\overline{MS}$ scheme. A $\overline{MS}$ result is obtained from the above rules by substituting $/91/$ $g \rightarrow g \cdot \exp \left[\frac{\epsilon}{2} (\gamma - \ln 4\pi) \right]$ before the UV poles are eliminated. Equivalently, the rules for the $\overline{MS}$ schemes at one loop is that one subtracts also the Euler constant γ and $\ln(4\pi)$ along with the UV poles.

To end this section we present some arguments motivating the factors (D10) and (D26) appearing at external legs of Feynman diagrams /93/. We feel these arguments necessary since renormalization is usually introduced within the on shell scheme where these factors reduce to $\sqrt{Z_3}$ and $\sqrt{Z_2}$, respectively.

For the presentation of the following arguments we best start from the Feynman rules stated after (D41):

Draw all possible topologically distinct diagrams without self energy insertions at the external legs, and associate with each spinor a factor

$$(D54) \quad \left[\frac{Z_2}{1 + i\Gamma_2} \right]^{1/2}$$

and with each polarization vector a factor

$$(D55) \quad \left[\frac{Z_3}{1 + \pi_R(0)} \right]^{1/2}$$

Starting from these version of Feynman rules has the advantage that we can avoid the procedure of amputating the external legs. The factors (D54) and (D55) can now be understood by verifying the following short cut

$$(D56) \quad \text{external line} = (\text{bare on shell propagator})^{1/2}$$

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Table 1: The flavour dependent electro weak coupling coefficients

$$g_{11}^f = Q_e^2 Q_f^2 + 2Q_e Q_f v v_f \text{Re} x_Z + (v^2 + a^2)(v_f^2 + a_f^2) |x_Z|^2$$

$$g_{12}^f = Q_e^2 Q_f^2 + 2Q_e Q_f v v_f \text{Re} x_Z + (v^2 + a^2)(v_f^2 - a_f^2) |x_Z|^2$$

$$g_{13}^f = 2Q_e Q_f v a_f \text{Im} x_Z$$

$$g_{14}^f = - \left[2Q_e Q_f v a_f \text{Re} x_Z + (v^2 + a^2) 2v_f a_f |x_Z|^2 \right]$$

$$g_{21}^f = Q_e^2 Q_f^2 + 2Q_e Q_f v v_f \text{Re} x_Z + (v^2 - a^2)(v_f^2 + a_f^2) |x_Z|^2$$

$$g_{22}^f = Q_e^2 Q_f^2 + 2Q_e Q_f v v_f \text{Re} x_Z + (v^2 - a^2)(v_f^2 - a_f^2) |x_Z|^2$$

$$g_{23}^f = g_{13}^f$$

$$g_{24}^f = - \left[2Q_e Q_f v a_f \text{Re} x_Z + (v^2 - a^2) 2v_f a_f |x_Z|^2 \right]$$

$$g_{31}^f = 2Q_e Q_f a v_f \text{Im} x_Z$$

$$g_{32}^f = g_{31}^f$$

$$g_{33}^f = - 2Q_e Q_f a a_f \text{Re} x_Z$$

$$g_{34}^f = - 2Q_e Q_f a a_f \text{Im} x_Z$$

$$g_{41}^f = - \left[2Q_e Q_f a v_f \text{Re} x_Z + 2v a (v_f^2 + a_f^2) |x_Z|^2 \right]$$

$$g_{42}^f = - \left[2Q_e Q_f a v_f \text{Re} x_Z + 2v a (v_f^2 - a_f^2) |x_Z|^2 \right]$$

$$g_{43}^f = g_{34}^f$$

$$g_{44}^f = 2Q_e Q_f a a_f \text{Re} x_Z + 4v a v_f a_f |x_Z|^2$$

Here

$$x_Z(s) = \frac{g m_Z^2 s}{s - m_Z^2 + i m_Z \Gamma_Z}$$

where m_Z and Γ_Z is the mass and total decay width of the Z^0 -boson, respectively; g is related to the Fermi weak coupling constant G_F by $g = G_F (8\pi\alpha\sqrt{2})^{-1}$.

Table 2: Angular coefficients $d_{K\alpha}(\theta, \kappa)$ arising in decomposing L.H

α	$K =$	A	B	C	D
1=U		$\frac{3}{8}(1 + \cos^2 \theta)$	$\frac{3}{8}\sin^2 \theta$	0	0
2=L		$\frac{3}{4}\sin^2 \theta$	$-\frac{3}{4}\sin^2 \theta$	0	0
3=T		$\frac{3}{4}\sin^2 \theta \cos 2\kappa$	$\frac{3}{4}(1 + \cos^2 \theta) \cos 2\kappa$	$\frac{3}{2}\cos \theta \sin 2\kappa$	0
4=I		$-\frac{3}{2\sqrt{2}}\sin 2\theta \cos \kappa$	$\frac{3}{2\sqrt{2}}\sin 2\theta \cos \kappa$	$\frac{3}{\sqrt{2}}\sin \theta \sin \kappa$	0
5		0	0	0	$\frac{3}{\sqrt{2}}\sin \theta \sin \kappa$
6		0	0	0	$\frac{3}{4}\cos \theta$
7		0	0	0	$-\frac{3}{\sqrt{2}}\sin \theta \cos \kappa$
8		$\frac{3}{2\sqrt{2}}\sin 2\theta \sin \kappa$	$-\frac{3}{2\sqrt{2}}\sin 2\theta \sin \kappa$	$\frac{3}{\sqrt{2}}\sin \theta \cos \kappa$	0
9		$-\frac{3}{4}\sin^2 \theta \sin 2\kappa$	$-\frac{3}{4}(1 + \cos^2 \theta) \sin 2\kappa$	$\frac{3}{2}\cos \theta \cos 2\kappa$	0

Table 3: Nonsingular contributions to 4 p.c. structure functions $\tilde{H}_i$. Entries denoted by $\leftarrow s$ ($\uparrow s$) are obtained from their left (upper) neighbours by interchanging 1 and 2. B_1 and B_2 are defined in the main text, $K \equiv \frac{1}{2\pi^2}$

K	$K \ln(1-x_1)$	$K \ln(1-x_2)$	$K r(1-x_2, 1-x_1)$
$7B_1 + \frac{(1-x_3)(x_1+x_2)}{(1-x_1)(1-x_2)}$	$-3\frac{x_2}{x_1} + \ln(1-x_1)^{B_1}$	$\leftarrow s$	$2B_1$
$7B_2 + \frac{2x_2(x_1+x_2)}{x_1(1-x_1)(1-x_2)}$	$\frac{2x_2^2}{x_1^2(1-x_1)(1-x_2)} + \ln(1-x_1)^{B_2}$	$\frac{-6}{(1-x_1)(1-x_2)} + \ln(1-x_2)^{B_2}$	$2B_2$
$\uparrow s$	$\frac{-6}{(1-x_1)(1-x_2)} + \ln(1-x_1)^{B_2}$	$\frac{2x_1^2}{x_2^2(1-x_1)(1-x_2)} + \ln(1-x_2)^{B_2}$	$\uparrow s$
$-2\frac{x_1+x_2}{x_1x_2}$	$-2\frac{(1-x_1)(1-x_2) + x_1x_2}{x_1^2(1-x_1)(1-x_2)}$	$\leftarrow s$	0

Table 4: Nonsingular contributions to 4 p.c. structure functions $\bar{H}_i$.

Notation $\leftarrow(\uparrow)$ as in table 3. $X \equiv (1-x_1)(1-x_2)(1-x_3)$.

	K	$K \ln(1-x_3)$
1	$7B_1 + \frac{(x_1-x_2)^2 + x_1 + x_2}{(1-x_1)(1-x_2)}$	$B_1 \ln(1-x_3) + \frac{2(1-x_3)(x_1+x_2-2x_1x_2)}{x_3(1-x_1)(1-x_2)}$
2	$7B_2 + 2 \frac{4x_1(1-x_2(1-x_2)) + x_2x_3(x_1+x_2)}{x_1x_3(1-x_1)(1-x_2)}$	$B_2 \ln(1-x_3) + 4 \frac{x_2^2x_3 - 2x_1x_2(1-x_1)(1-x_2) + 2(1+x_2)X}{x_3^2(1-x_1)^2(1-x_2)^2}$
3	$\uparrow s$	$\uparrow s$
4	$-\left[x_1x_2x_3(x_3-6(1-x_3)) - 8x_1x_2(x_1^2+x_2^2) \right. \\ \left. + x_1x_3(x_3^2-x_1^2) + x_2x_3(x_3^2-x_2^2) \right] \\ \frac{1}{2x_1x_2x_3(1-x_1)(1-x_2)}$	$\frac{x_3^2(1-x_3) + (1-x_1)(1-x_2)(x_1^2+x_2^2-x_3)}{4x_3^2(1-x_1)^2(1-x_2)^2}$

ple 4 continued

$\text{Kln}(1-x_2)$	$\text{Kln}(1-x_1)$	$\text{Kr}(1-x_3, 1-x_2)$	$\text{Kr}(1-x_3, 1-x_1)$
$\frac{x_1(1+x_1-2x_3)}{x_2(1-x_1)}$	$\frac{2x_2^2(2x_1+1-x_2)}{x_1^2(1-x_1)(1-x_2)^2}$	$\frac{x_2^2(1-x_1)+x_1x_3(1-x_3)}{2(1-x_1)^2(1-x_2)}$	$\frac{x_2^2+2(1-x_2)^2}{4(1-x_1)(1-x_2)^3}$
$\frac{x_1^2(2x_2+1-x_1)}{x_2^2(1-x_1)^2(1-x_2)}$	$\frac{1+x_2-2x_3}{2(1-x_1)(1-x_2)^2}$	$\frac{x_1^2+2(1-x_1)^2}{4(1-x_1)^3(1-x_2)}$	$\frac{(1-x_3)^2+(1-x_2)^2}{4(1-x_1)(1-x_2)^3}$
$\frac{(1-x_3)^2+(1-x_3)x_2-2(1-x_1)^2(1-x_2)}{2x_2^2(1-x_1)^2(1-x_2)}$	$\frac{x_1^2+x_2-1}{4(1-x_1)^3(1-x_2)}$		

TABLE 5
The interchange table relating the graphs for $e^+e^- \rightarrow q\bar{q}GG$

Permu- tation of first row	Group weight		Class A C_F^2	Class B $C_F(C_F - \frac{1}{2}N_C)$	C_I	Class C $C_F N_C$	C_G
(3↔4) (1↔2) (1↔2), (3↔4)			B11 B32 B21 B22	B42 B52 B53 B41	B71 B72	B82 B87	B77 B88
			B44 B65 B54 B55	B51 B62	B74 B75	B85 B88	
			B64 B66	B61	B84 B86	B76	
			B31 B33	B43 B63	B81 B83	B73	

TABLE 6
The interchange table relating the graphs for $e^+e^- \rightarrow \bar{q}q\bar{q}q$

Permu- tation of first row	Group weight		Class D $C_F T_R$		Class E $C_F(C_F - \frac{1}{2}N_C)$		Class F C_F					
	A77	A88	A87	A83	A76	A73	A86	A84	A75	A81	A82	A53
(1↔2)	A55	A66	A65	A61	A85	A51		A62		A63	A64	A71
(3↔4)	A33	A44	A43	A74	A32		A42		A31	A54		
(1↔2), (3↔4)	A11	A22	A21	A52	A41					A72		

Table 7: Coefficient-functions for the gluon-polarization

$$F_{\text{lin}}^1(x_1, x_2) = -2(1-x_3) \frac{x_1+x_2+14}{(1-x_1)(1-x_2)}$$

$$F_{\text{sum}}^1(x_1, x_2) = 2 \frac{-7x_1x_2(x_1^2+x_2^2) - (1-x_3)^2(x_1+x_2)}{(1-x_1)(1-x_2)x_1x_2}$$

$$F_{\text{cir}}^1(x_1, x_2) = 2(x_1-x_2) \cdot \frac{7x_1^2x_2 + x_1^2 + 7x_1x_2^2 + 2x_1x_2 - 2x_1 + x_2^2 - 2x_2 + 1}{(1-x_1)(1-x_2)x_1x_2}$$

$$F_{\text{lin}}^2(x_1, x_2) = 0$$

$$F_{\text{sum}}^2(x_1, x_2) = 2 \frac{3x_1x_2 + x_1 + x_2 - 1}{x_2^2}$$

$$F_{\text{cir}}^2(x_1, x_2) = -F_{\text{sum}}^2(x_1, x_2)$$

$$F_{\text{lin}}^3(x_1, x_2) = -8(1-x_3) \frac{1}{(1-x_1)(1-x_2)}$$

$$F_{\text{sum}}^3(x_1, x_2) = -4 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

$$F_{\text{cir}}^3(x_1, x_2) = 4 \frac{x_1^2 - x_2^2}{(1-x_1)(1-x_2)}$$

$$F_{\text{lin}}^4(x_1, x_2) = 2(1-x_3) \frac{x_1^2 - 2x_1x_2 - 16x_1 + x_2^2 - 16x_2 + 32}{(1-x_1)(1-x_2)x_3}$$

$$F_{\text{sum}}^4(x_1, x_2) = -2 \frac{7x_1x_2(x_1+x_2-2)(x_1^2+x_2^2)}{x_1x_2x_3(1-x_1)(1-x_2)} - 2 \frac{[x_1^4+2x_1^3x_2-4x_1^3+3x_1^2x_2^2-8x_1^2x_2+5x_1^2+3x_1x_2-2x_1] + [1 \leftrightarrow 2]}{x_1x_2x_3(1-x_1)(1-x_2)}$$

$$F_{\text{cir}}^4(x_1, x_2) = -2 \frac{7x_1^2x_2+x_1^2+7x_1x_2^2+4x_1x_2-2x_1+x_2^2-2x_2+1}{x_1x_2(1-x_1)(1-x_2)}$$

$$F_{\text{lin}}^5(x_1, x_2) = -8(1-x_3) \frac{1}{x_3^2}$$

$$F_{\text{sum}}^5(x_1, x_2) = 8(1 - \frac{1}{x_3^2})$$

$$F_{\text{cir}}^5(x_1, x_2) = 0$$

$$F_{\text{lin}}^6(x_1, x_2) = 0$$

$$F_{\text{sum}}^6(x_1, x_2) = 2 \frac{-3x_1x_2 - x_1 - 2x_2^2 + x_2 + 1}{x_2^2}$$

$$F_{\text{cir}}^6(x_1, x_2) = -F_{\text{sum}}^6(x_1, x_2)$$

$$F_{\text{lin}}^7(x_1, x_2) = 8(1-x_3) \frac{1}{(1-x_1)(1-x_2)}$$

$$F_{\text{sum}}^7(x_1, x_2) = 4 \frac{(1-x_1)^2 - 2x_2x_3 + 2x_2}{(1-x_1)(1-x_2)}$$

$$F_{\text{cir}}^7(x_1, x_2) = 4 \frac{x_1 + 2x_2 - 1}{1 - x_2}$$

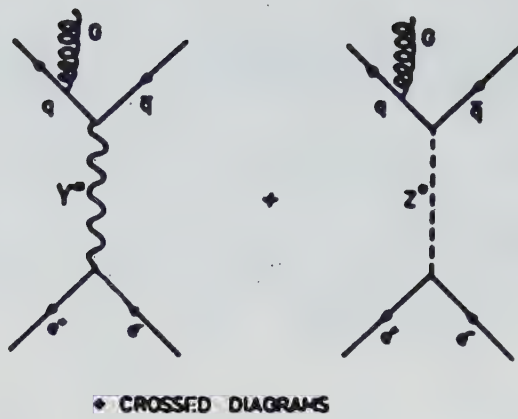


Fig. 1. $q\bar{q}G$ production in e^+e^- collisions via electromagnetic and weak interactions

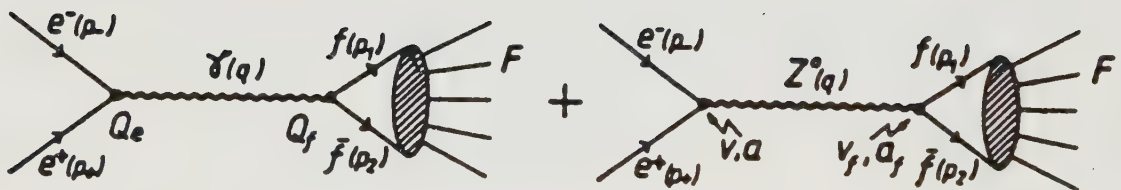


Fig.2 One-photon and one - Z^0 exchange diagram for $e^+e^- \rightarrow f\bar{f} \rightarrow F$.

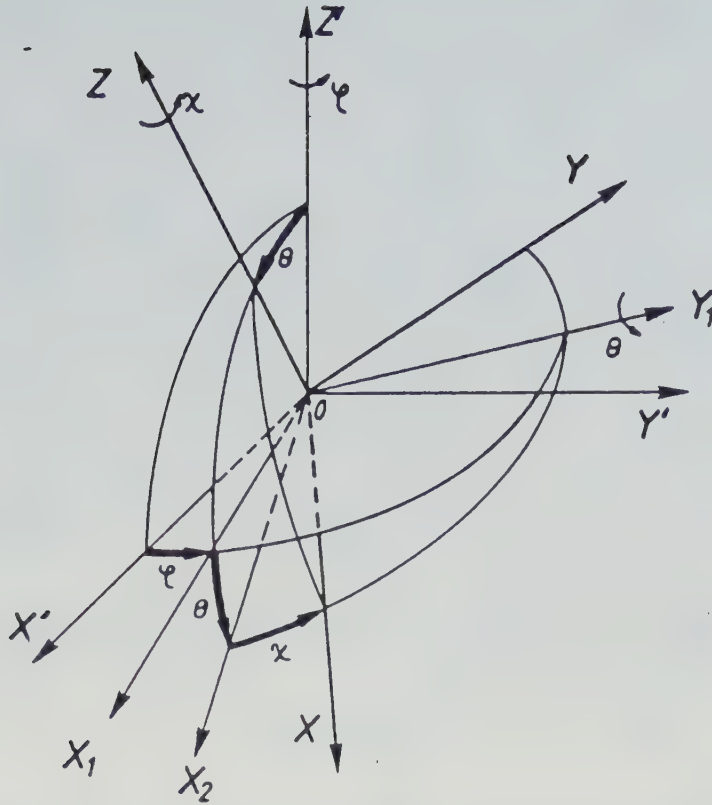


Fig.3 Definition of Euler angles ϕ , θ and χ . Oz' points into the direction of the electron beam and Ox' into the direction of the Lorentz force. The frame $Oxyz$ is fixed to the final state. Any two noncolinear final state momenta can be chosen to define its relative orientation.

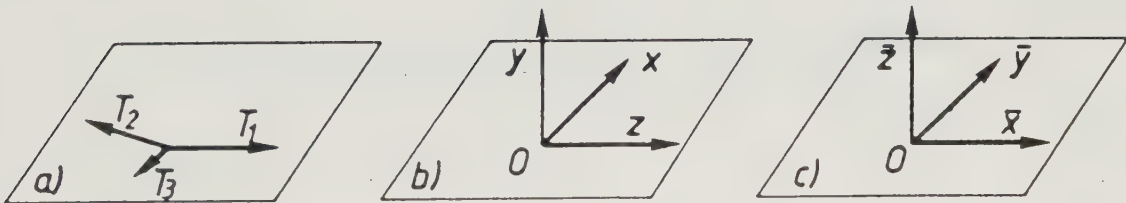


Fig.4 Natural choices of frame $Oxyz$ for three-jet final states. (a) Three-jet event ($T_1 = 2E_1/Q$). (b) Helicity frame with Oz along the jet axis (i.e., the axis of the most energetic jet) and Ox pointing into the half-plane of the second most energetic jet. (c) Transversity frame with $O\bar{x}$ along the jet axis and $O\bar{y}$ pointing into the half-plane of the second most energetic jet. Here the quantization axis $O\bar{z}$ is the normal to the event plane.

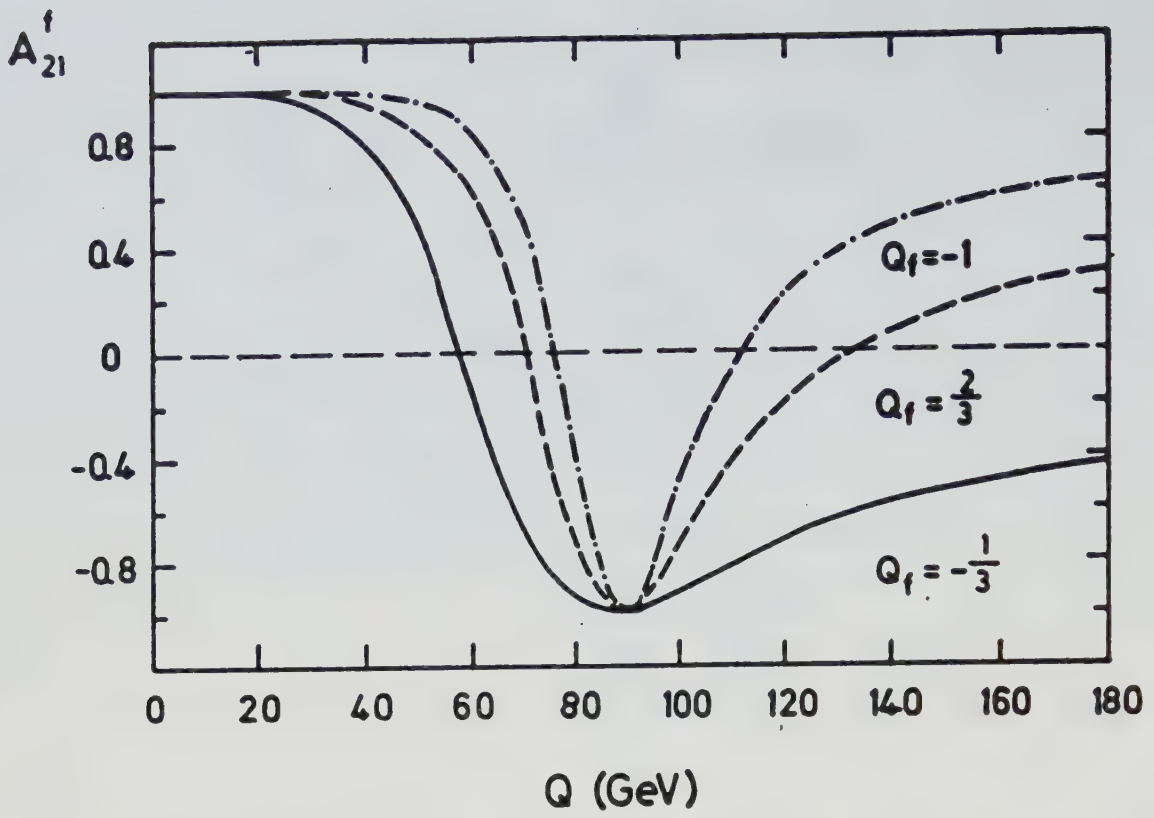


Fig.5 Azimuthal asymmetry A_{21}^f ($-\cos 2\phi$) for leptons and quarks as predicted by the Weinberg-Salam model and $\sin^2 \theta_W = 0.23$.

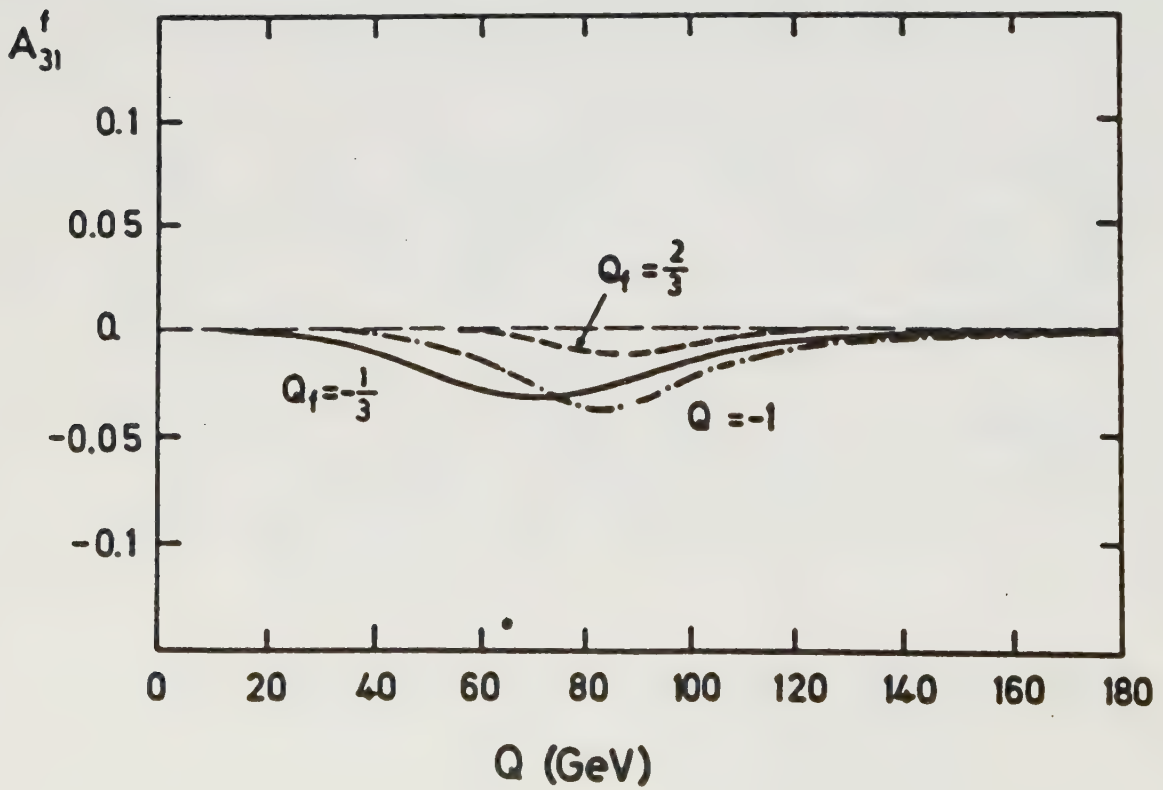


Fig.6 Azimuthal asymmetry A_{31}^f ($-\sin 2\phi$).

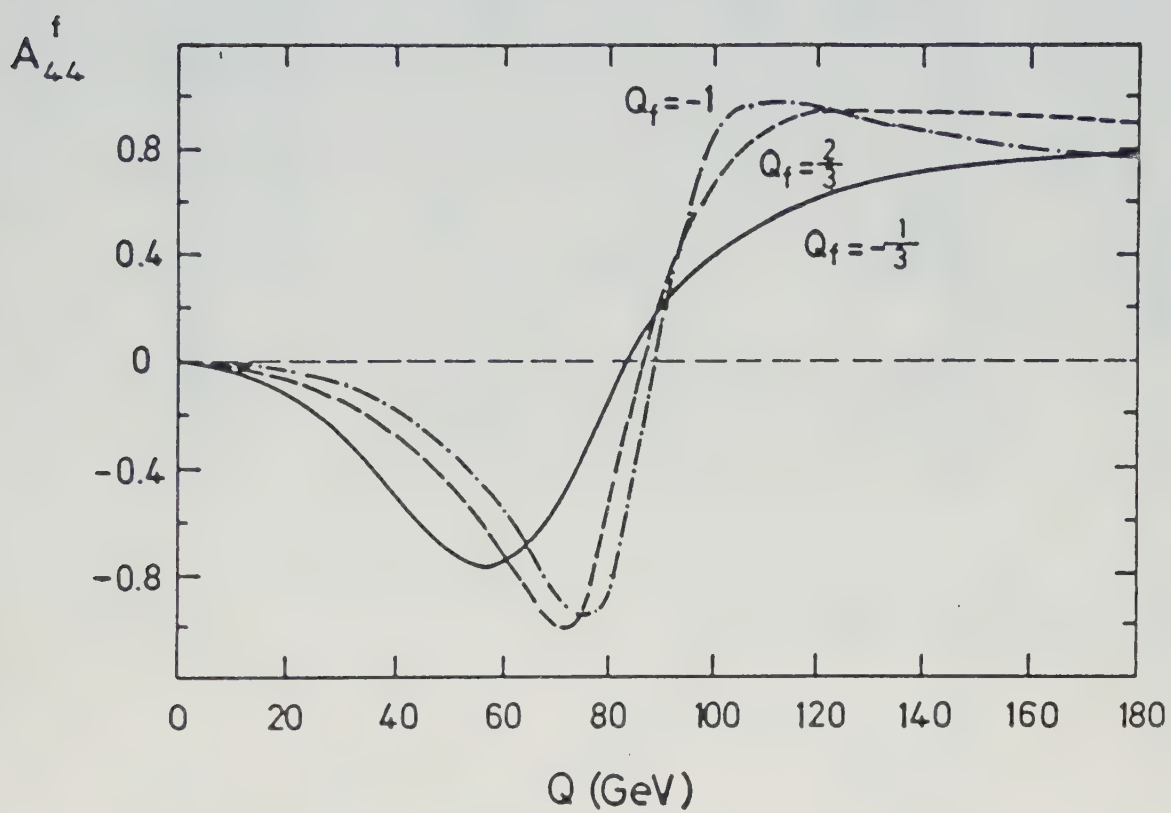
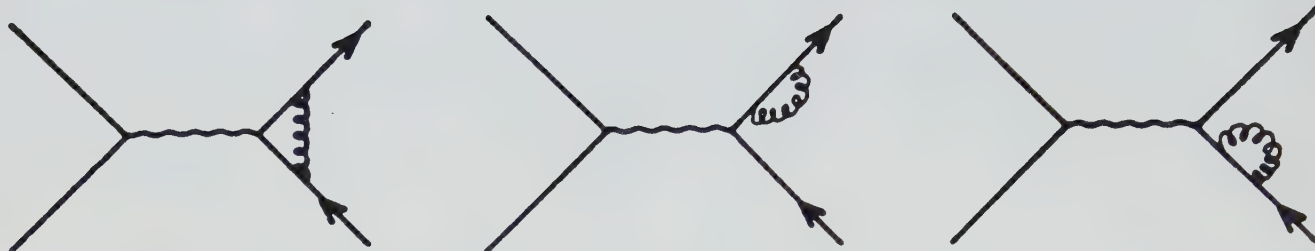


Fig.7 Forward-backward asymmetry A_{44}^f ($-\cos\theta$).

a)



b)

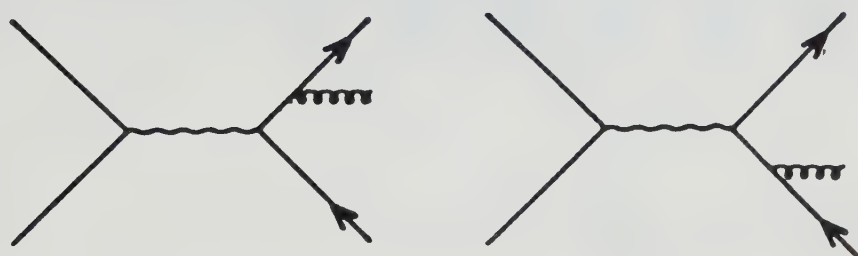


Fig. 8: Feynman diagrams for a) $q\bar{q}$ and b) $q\bar{q}g$ production in the order α_s

DIRECT DRESSING

PARTIAL FRACTIONED DRESSING

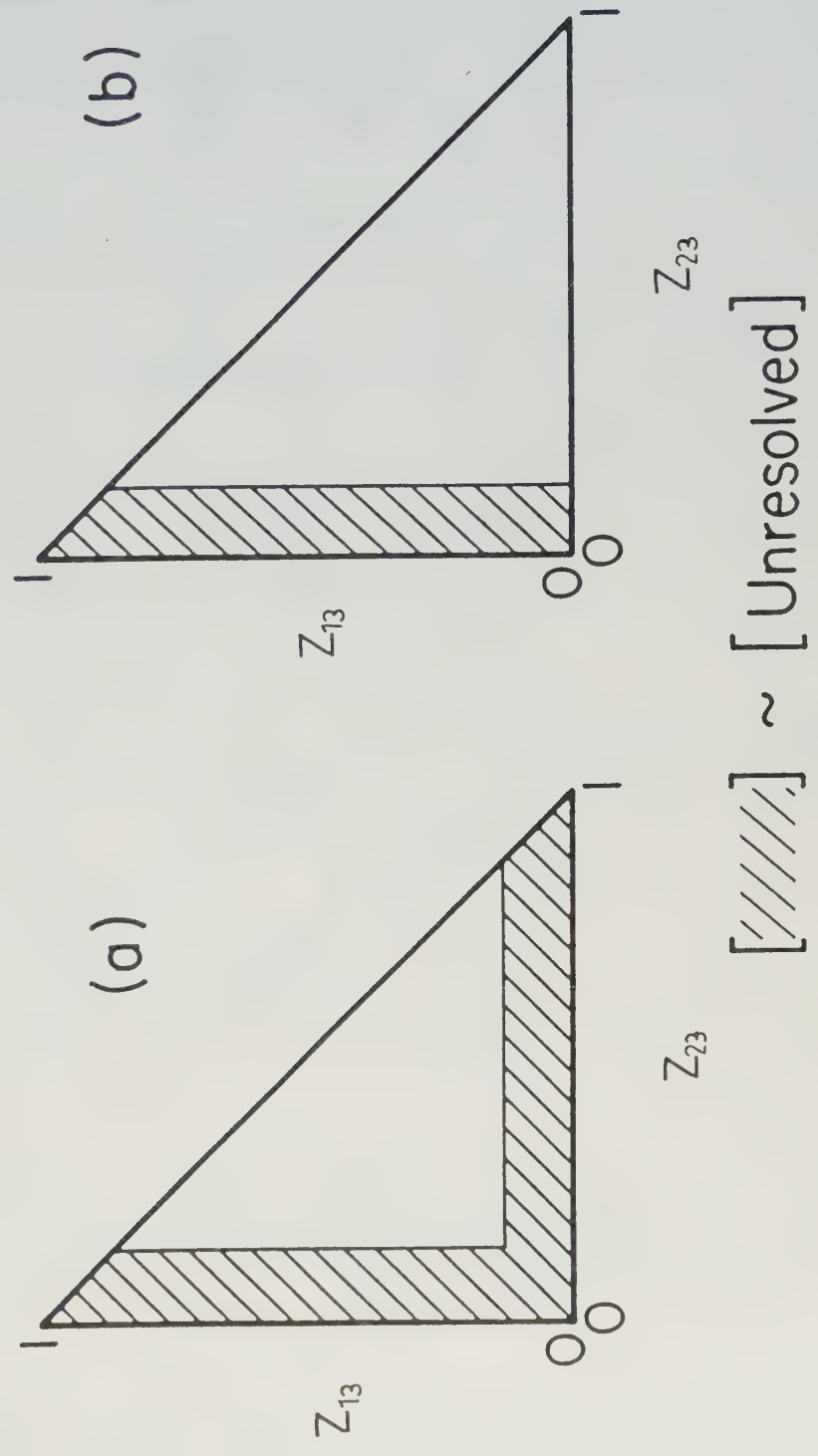


Fig. 9 : Unresolved regions of 3-parton phase space in the Direct Dressing and Partial Fractioned Dressing schemes.

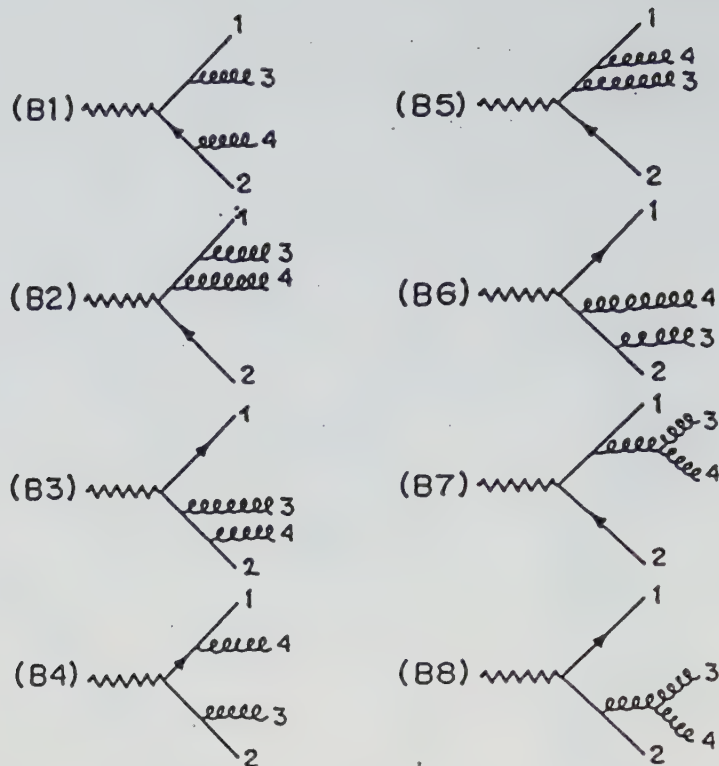


Fig.10 . Diagrams contributing to the process $\gamma^* \rightarrow q\bar{q}GG$.

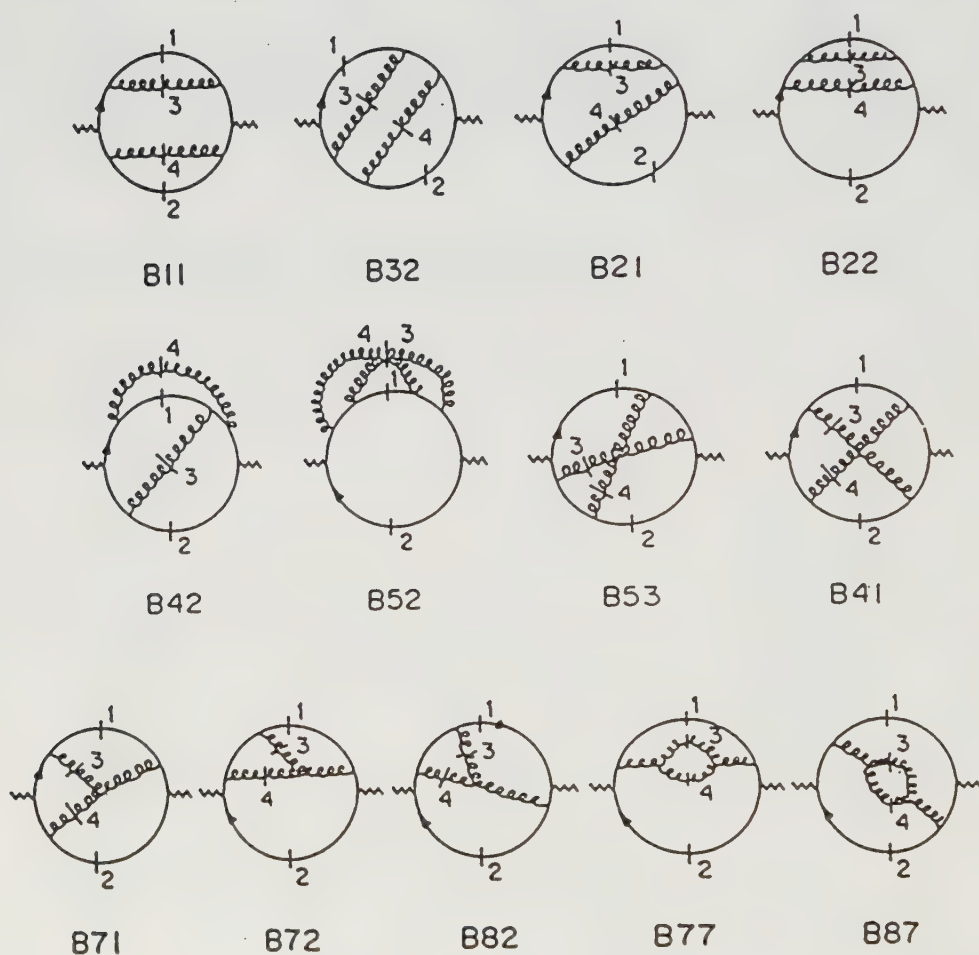


Fig.11 Transition probabilities for the process $\gamma^* \rightarrow q\bar{q}GG$. On-shell particles are indicated by the short cutting line and the numbers refer to the momentum labels. All the other transition probabilities can be obtained by permutation of momentum labels.

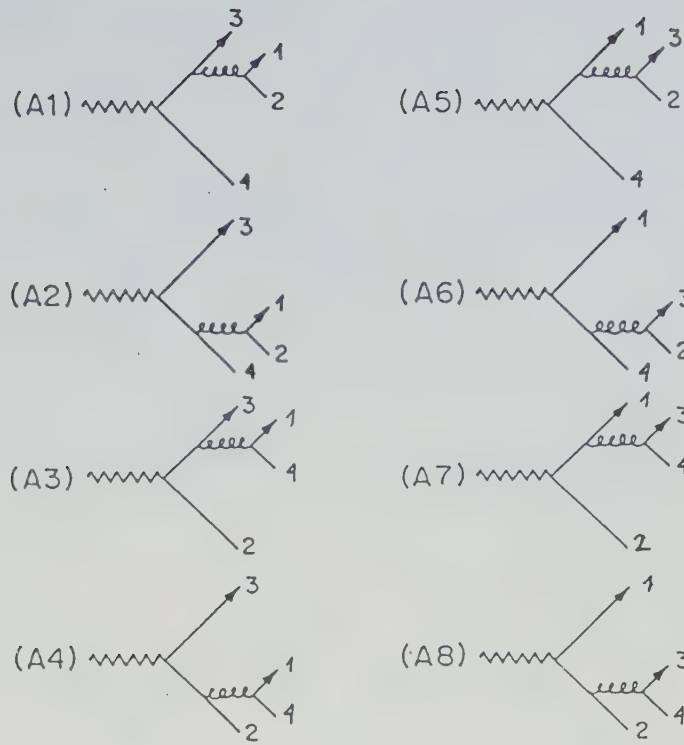


Fig.12: Diagrams contributing to the process $\gamma^* \rightarrow q\bar{q}q\bar{q}$. Diagrams which differ only by the exchange of identical final state fermions have a relative minus sign.

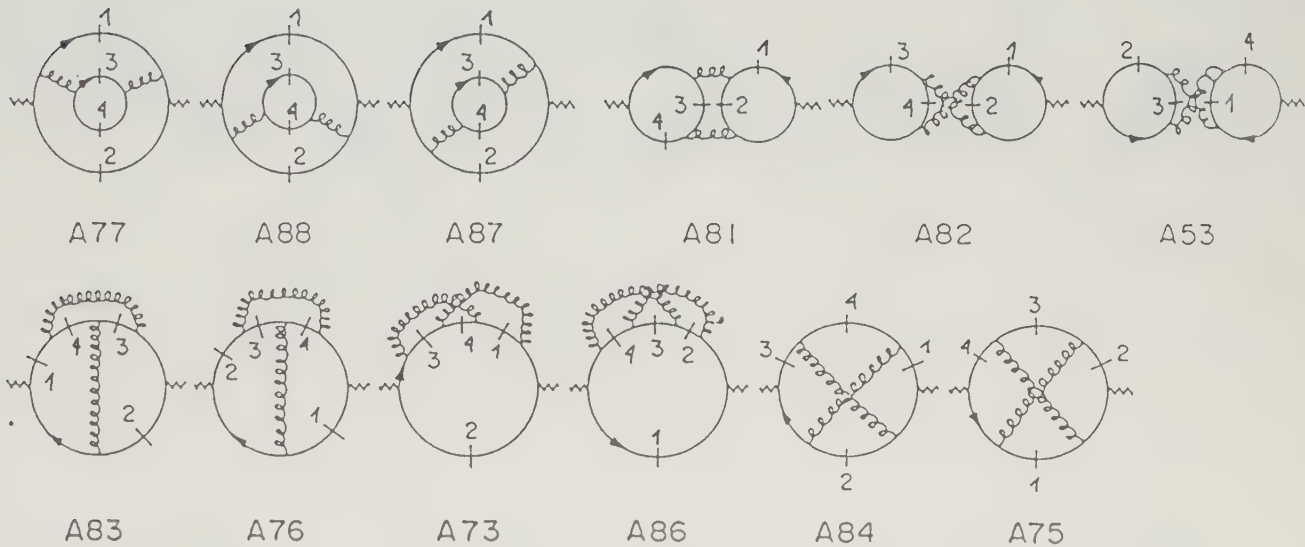


Fig.13: Transition probabilities for the process $\gamma^* \rightarrow q\bar{q}q\bar{q}$. On-shell particles are indicated by a short cutting line and the numbers refer to the momentum labels. All other transition probabilities can be obtained by exchange of momentum labels.

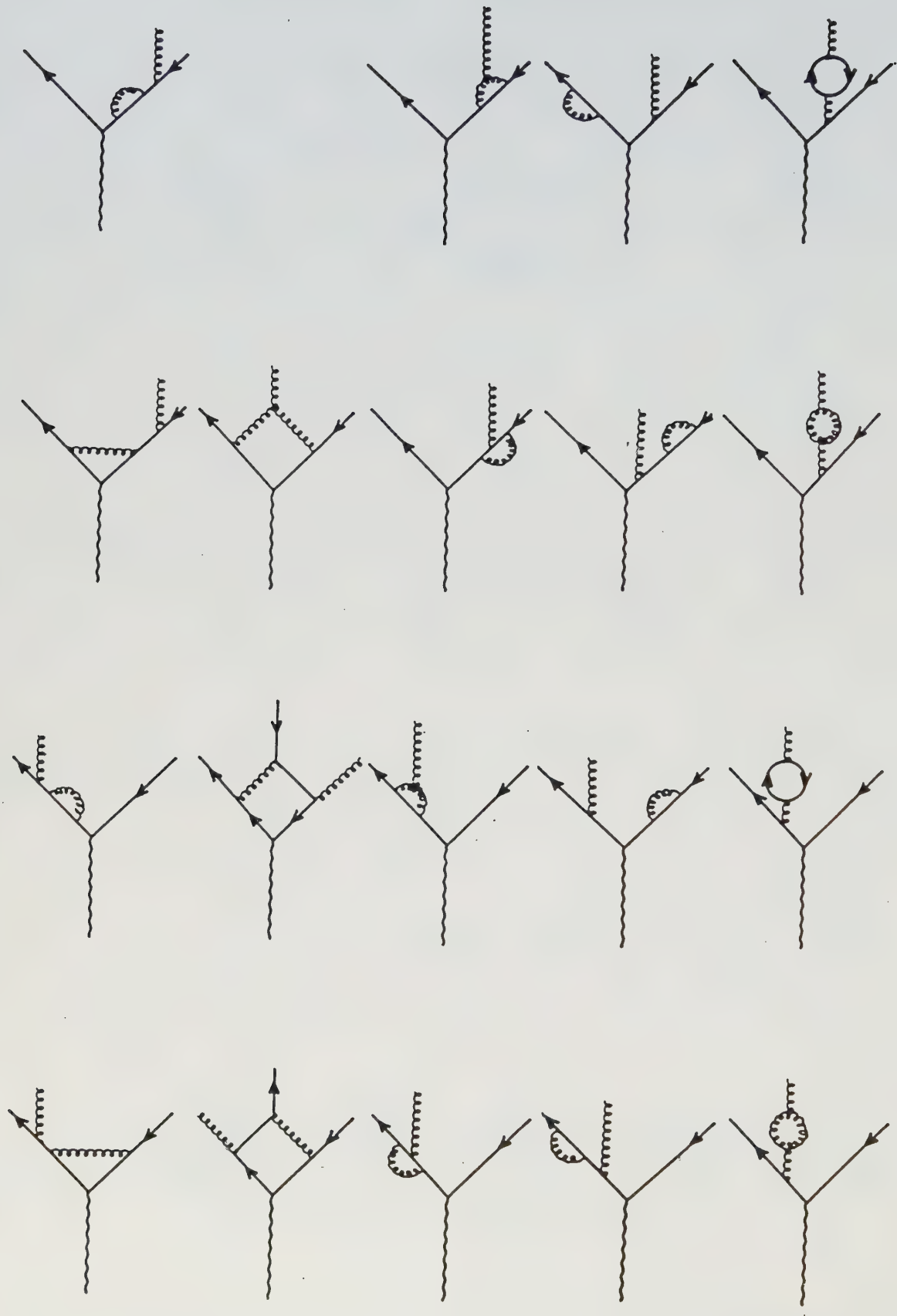


Fig.14: Diagrams with $q\bar{q}g$ in the final state to order α_s^2

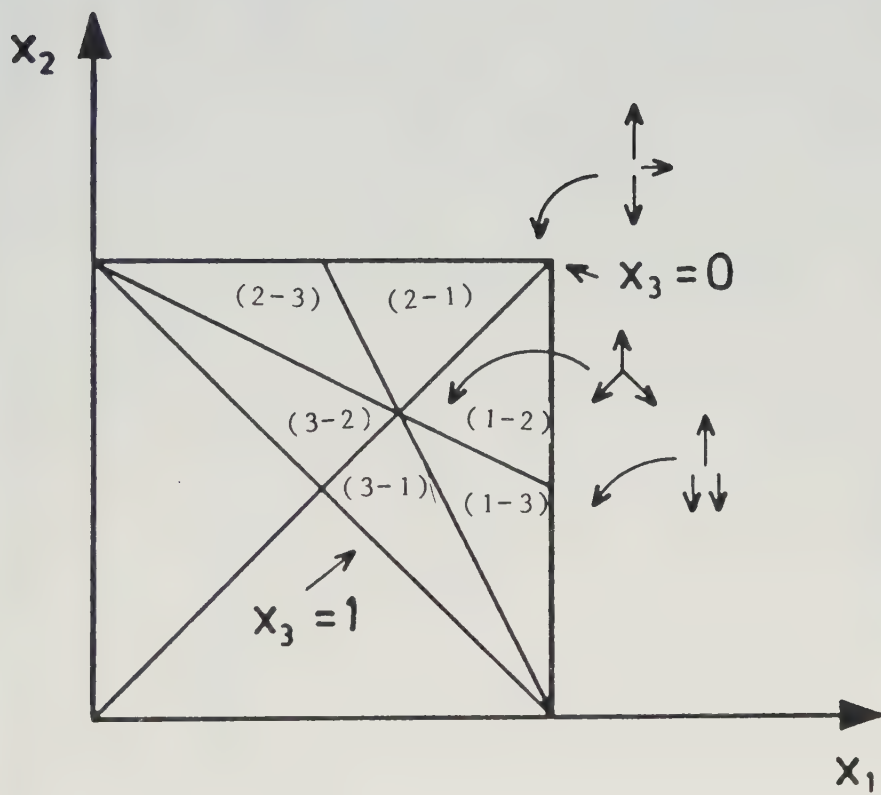


Fig.15: Phase space diagram for $e^+e^- \rightarrow q\bar{q}g$ with regions ordered according to magnitude of x_1 , x_2 and x_3

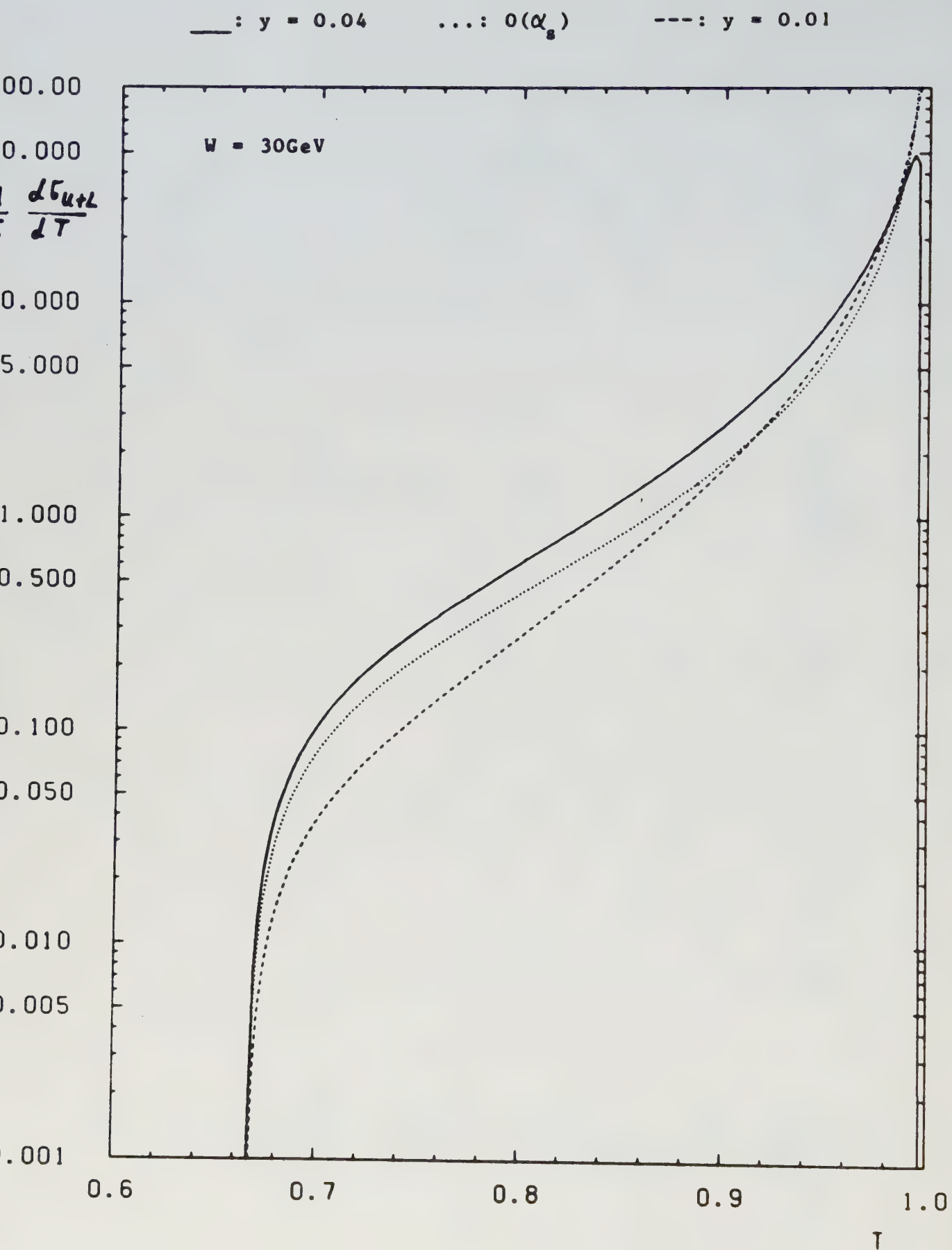


Fig.16 The $O(\alpha_s^2)$ 3-jet cross section $(1/\sigma)d\sigma_{U+L}/dT$ for $y = 0.04$ and $y = 0.01$ together with the corresponding Born cross section $(O(\alpha_s))$ as a function of thrust T for $\alpha_s = 0.16$.

σ is defined in (7.2.4).

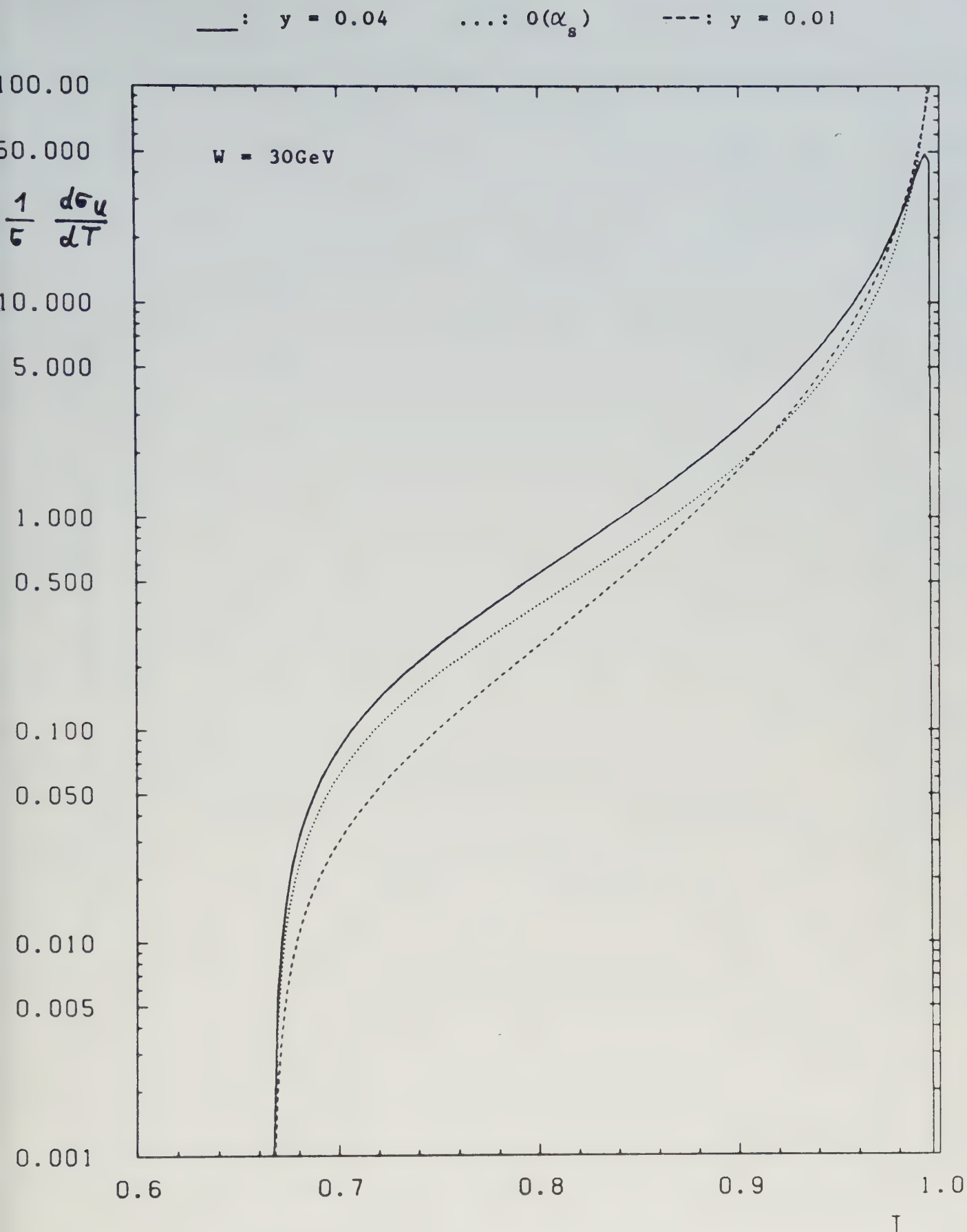


Fig.17 The $O(\alpha_s^2)$ 3-jet cross section $(1/\sigma)d\sigma_U/dT$ for $y = 0.04$ and $y = 0.01$ together with the corresponding Born cross section $(O(\alpha_s))$ as a function of thrust T for $\alpha_s = 0.16$.
 σ is defined in (7.2.4).

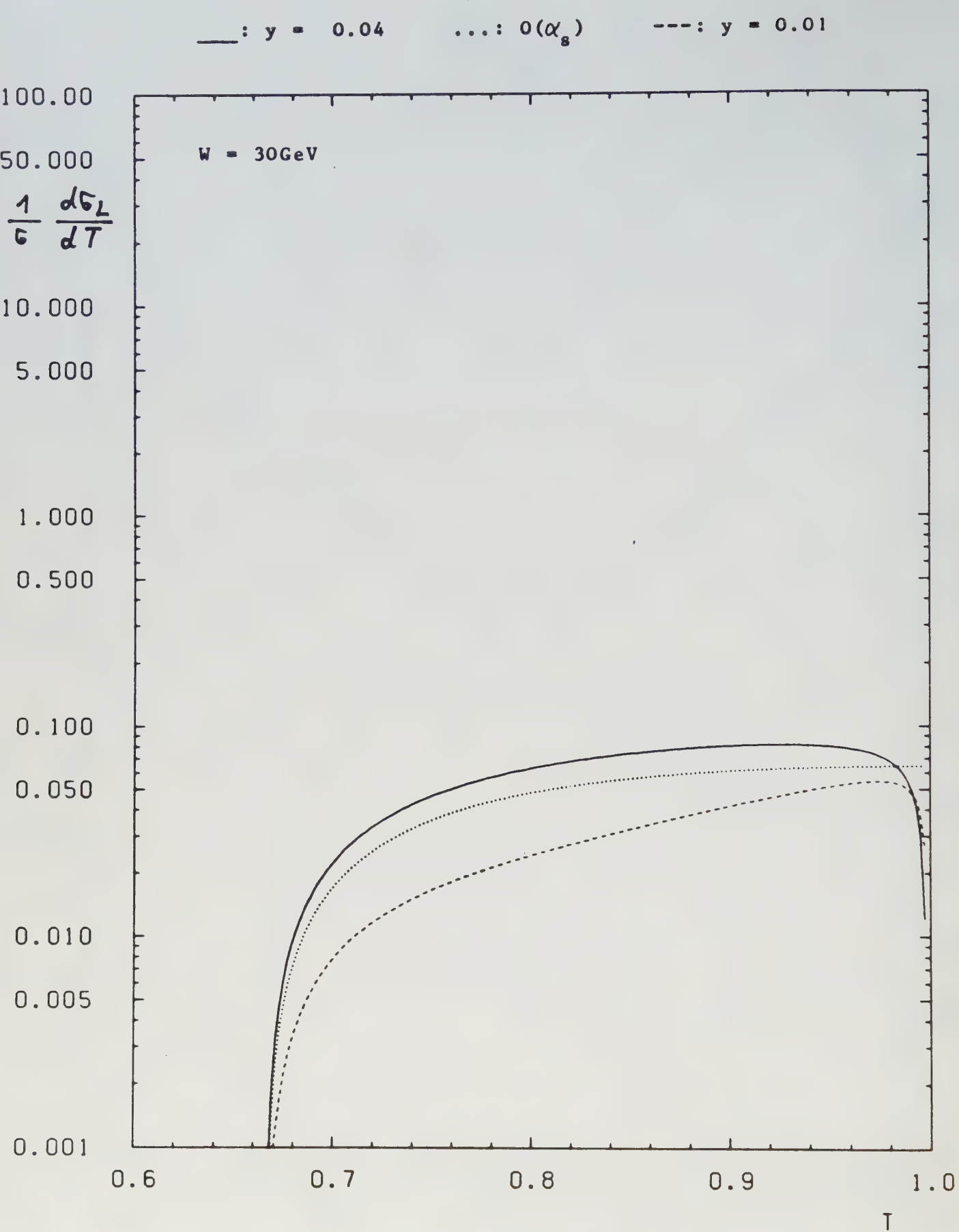


Fig.18 The $O(\alpha_s^2)$ 3-jet cross section $(1/\sigma)d\sigma_L/dT$ for $y = 0.04$ and $y = 0.01$ together with the corresponding Born cross section $(O(\alpha_s))$ as a function of thrust T for $\alpha_s = 0.16$.
 σ is defined in (7.2.4).

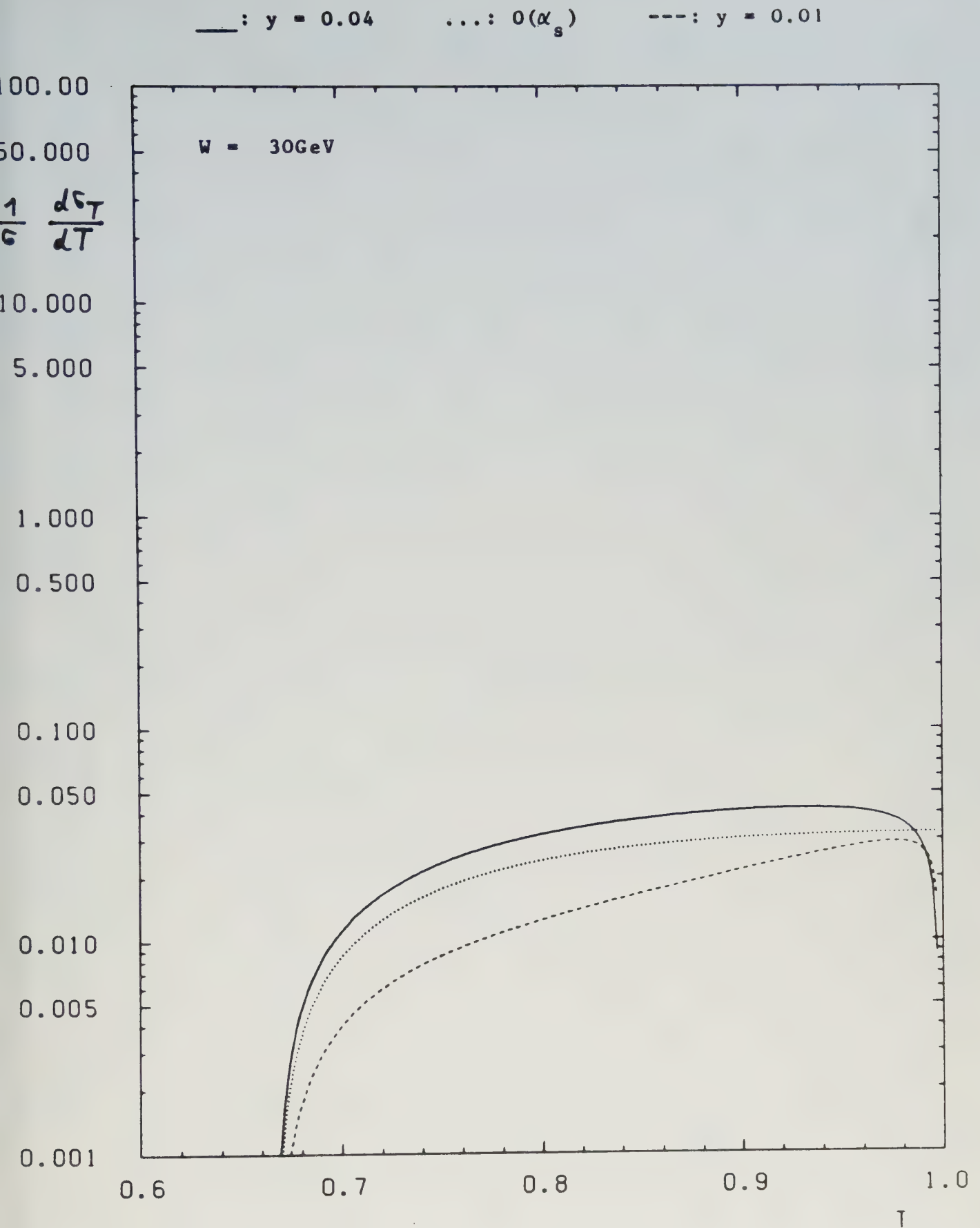


Fig.19 The $O(\alpha_s^2)$ 3-jet cross section $(1/\sigma)d\sigma_T/dT$ for $y = 0.04$ and $y = 0.01$ together with the corresponding Born cross section $(O(\alpha_s))$ as a function of thrust T for $\alpha_s = 0.16$. σ is defined in (7.2.4).

—: $y = 0.04$...: $O(\alpha_s^2)$ ---: $y = 0.01$

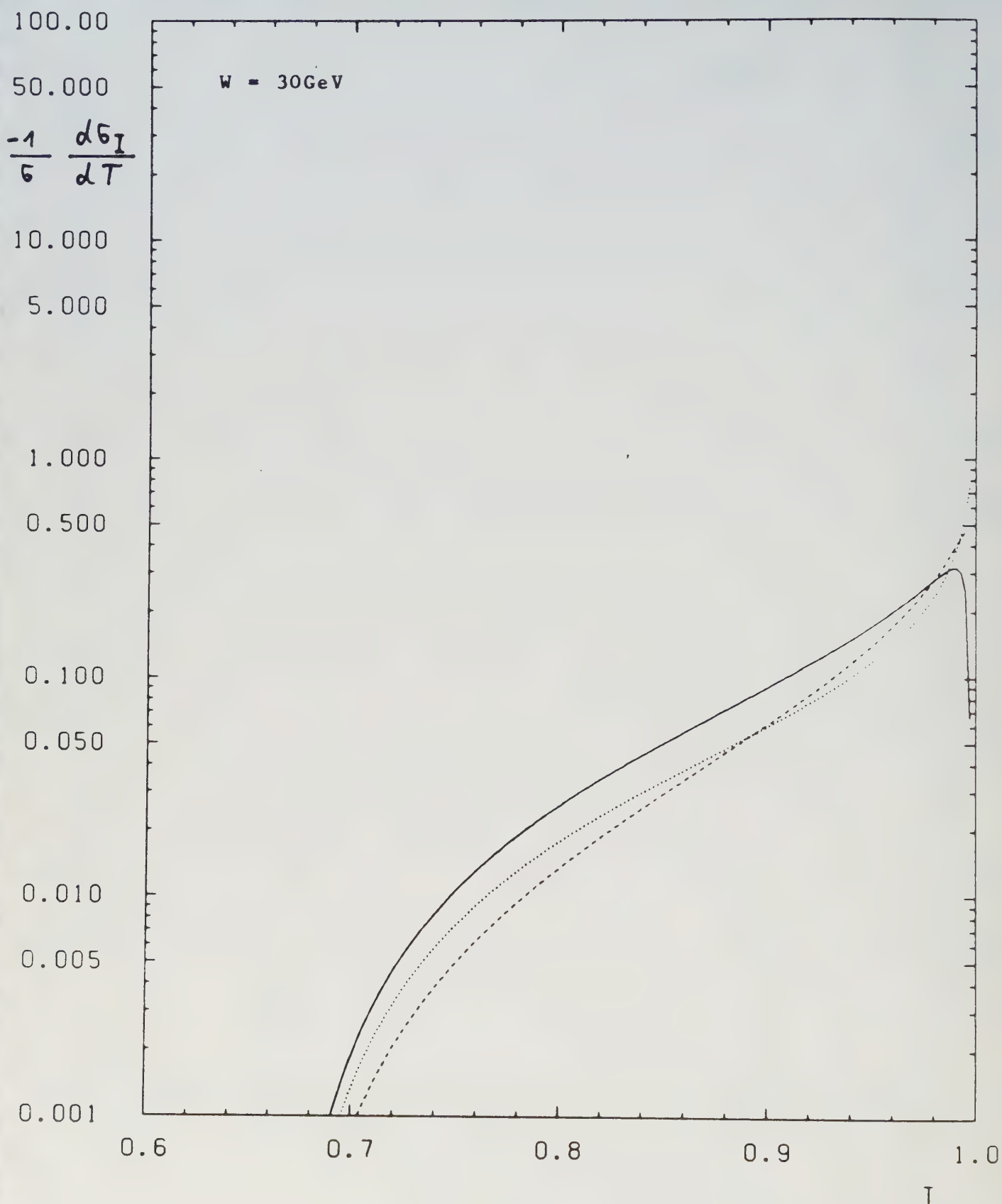


Fig.20 The $O(\alpha_s^2)$ 3-jet cross section $(-1/\sigma)d\sigma_I/dT$ for $y = 0.04$ and $y = 0.01$ together with the corresponding Born cross section $O(\alpha_s)$ as a function of thrust T for $\alpha_s = 0.16$.
 σ is defined in (7.2.4).

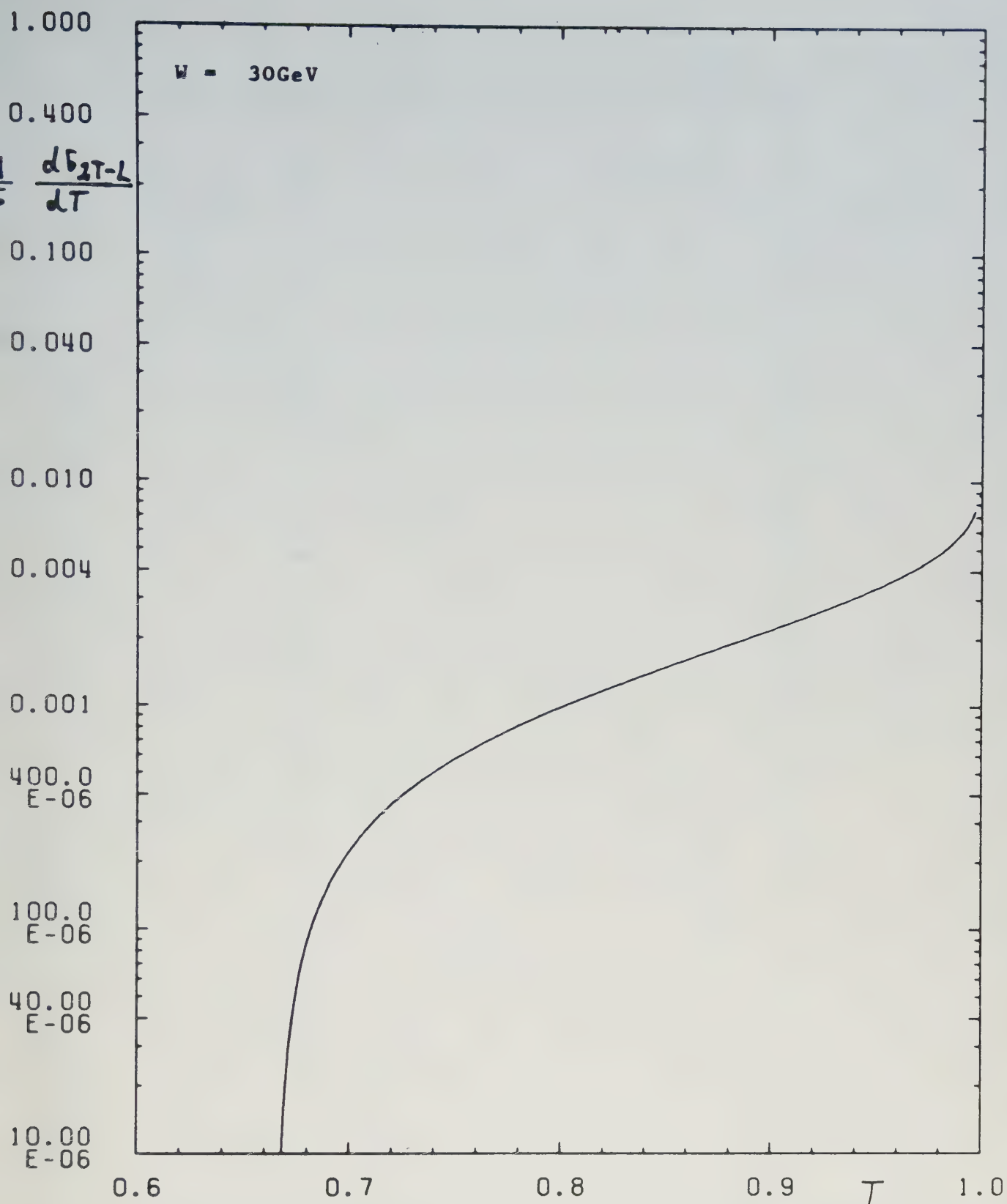


Fig.21 The $O(\alpha_s^2)$ 3-jet cross section $(1/\sigma)d\sigma_{2T-L}/dT$
as a function of thrust T for $\alpha_s = 0.16$.
 σ is defined in (7.2.4).

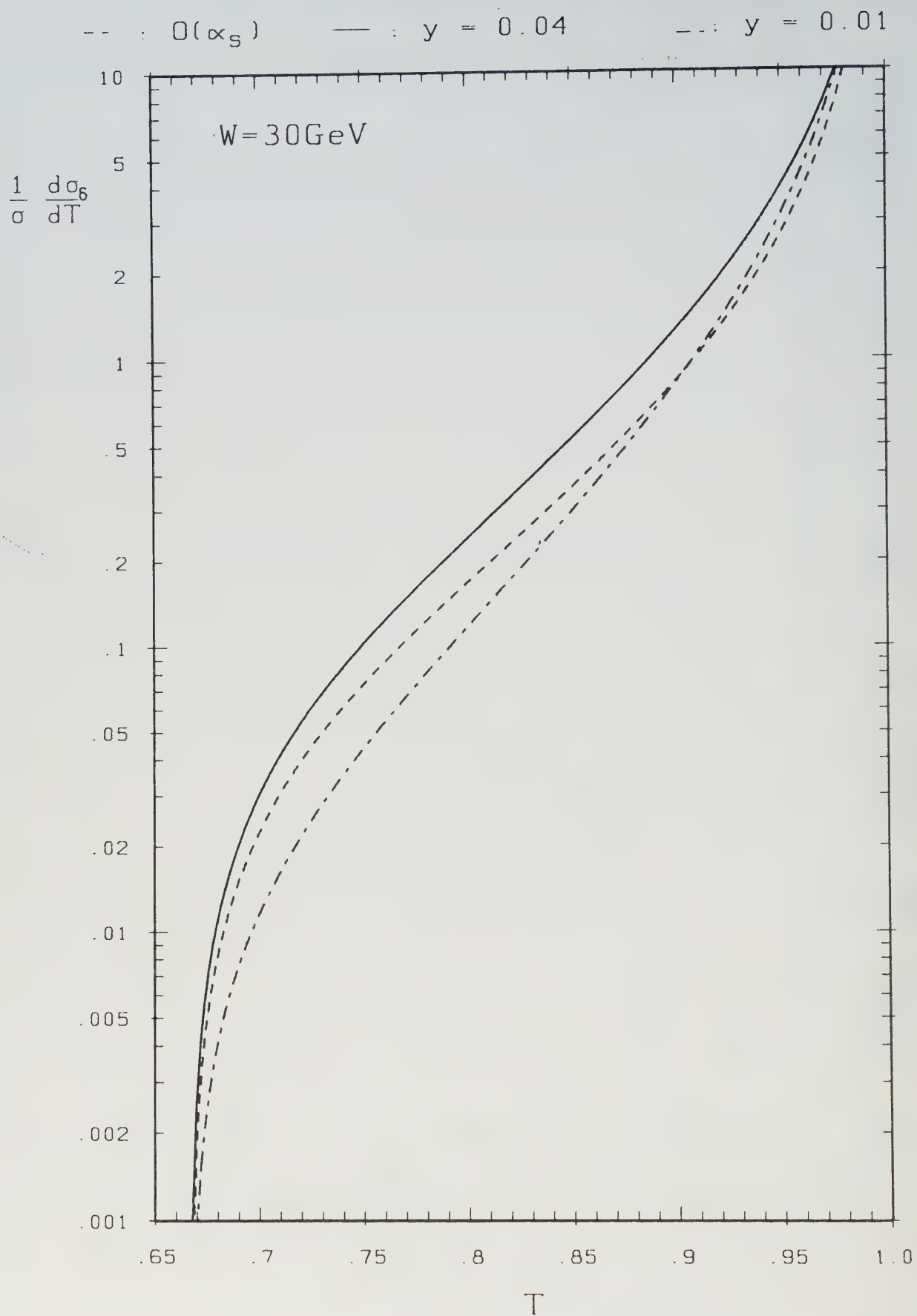


Fig.22 The p.v. $O(\alpha_s^2)$ 3-jet cross section $(1/\sigma)d\sigma_6/dT$ for $y = 0.04$ and $y = 0.01$ together with the corresponding Born cross section $O(\alpha_s)$ as a function of thrust T for $\alpha_s = 0.16$. σ is defined in (7.2.4). The thrust axis is the quark momentum.

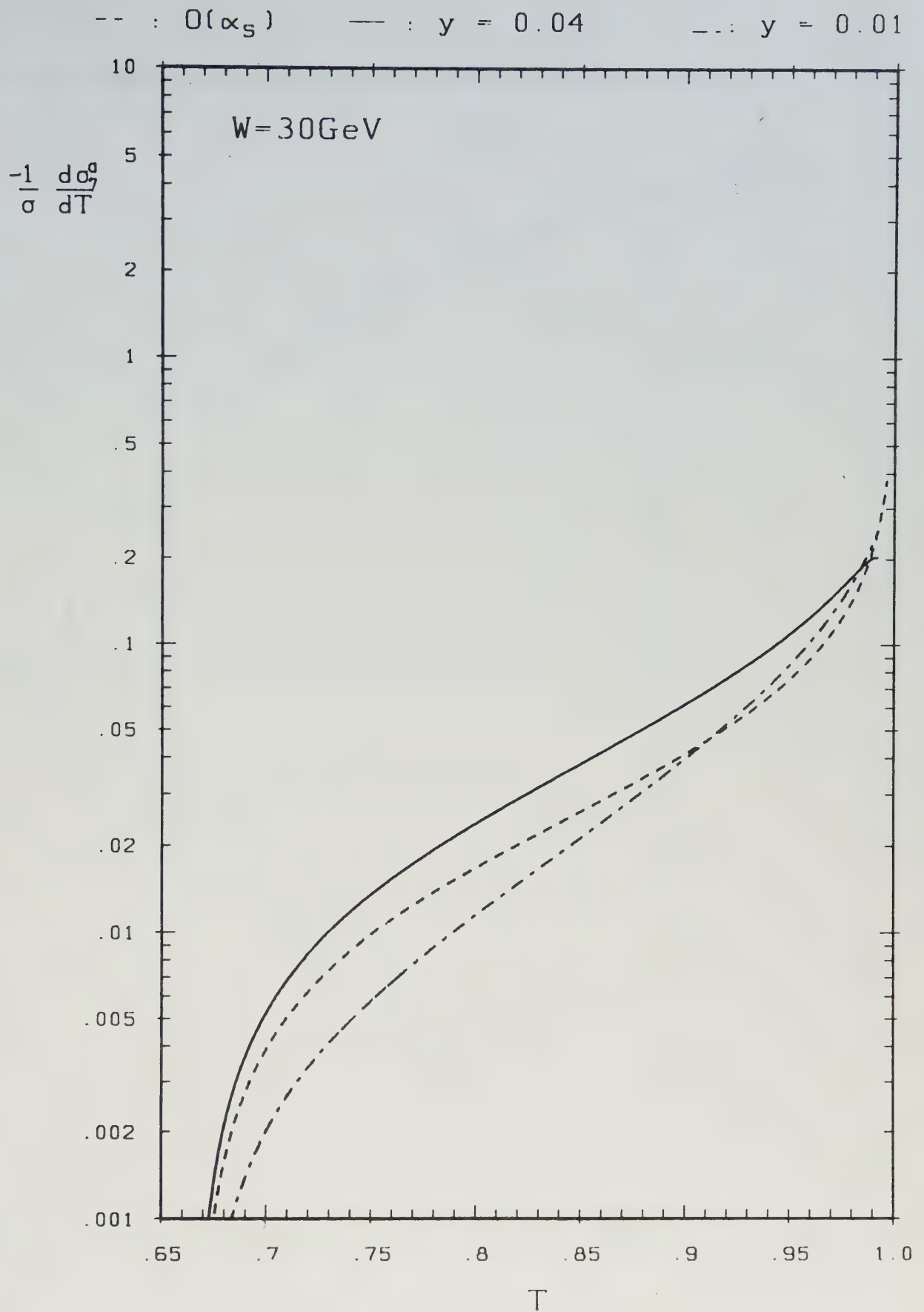


Fig.23 The p.v. $O(\alpha_s^2)$ 3-jet cross section $(-1/\sigma)d\sigma_7/dT$ for $y = 0.04$ and $y = 0.01$ together with the corresponding Born cross section $(0(\alpha_s))$ as a function of thrust T for $\alpha_s = 0.16$. σ is defined in (7.2.4). The thrust axis is the quark momentum. The $\bar{x}$ -axis is in the direction of the antiquark.

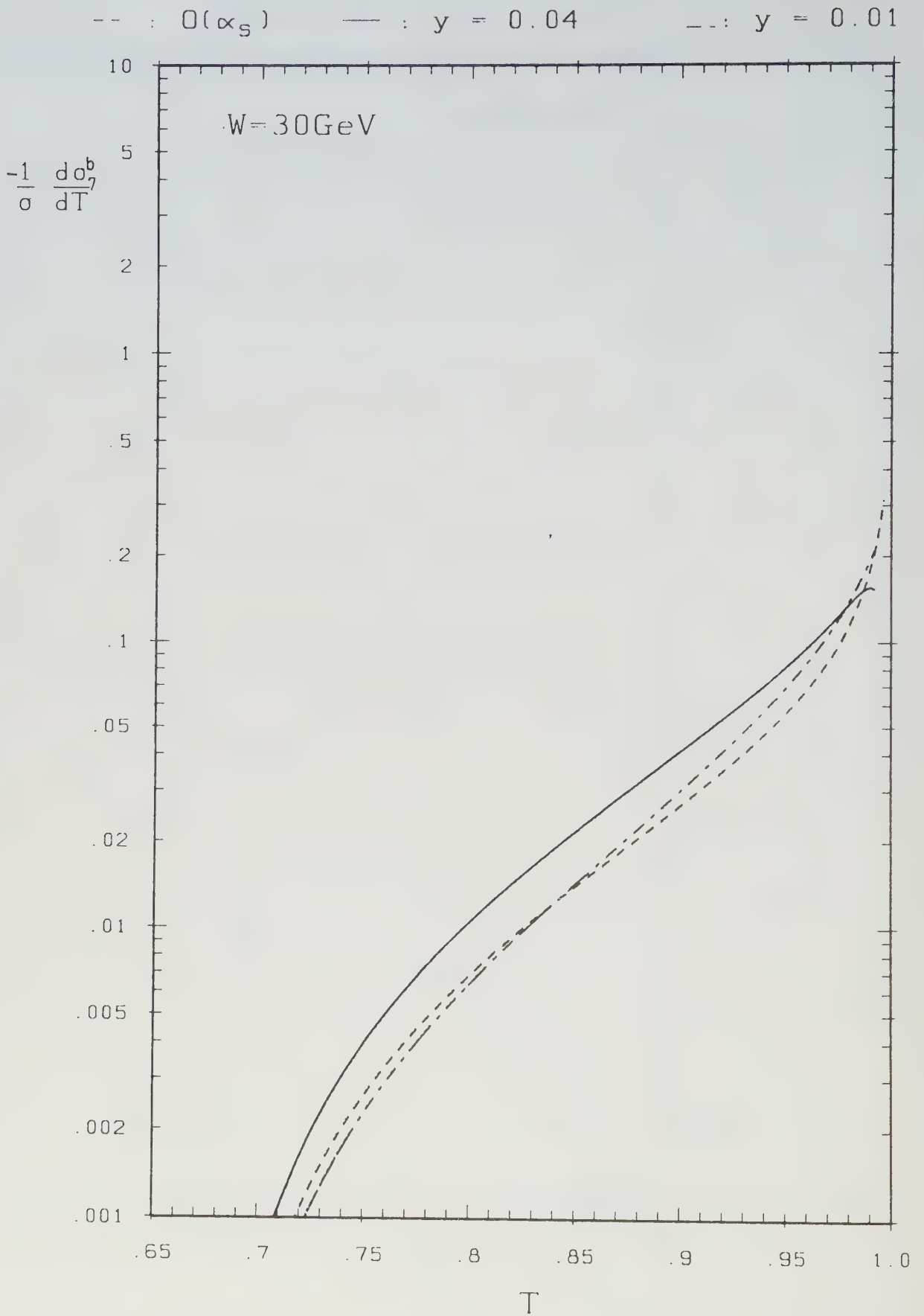


Fig.24 The p.v. $O(\alpha_s^2)$ 3-jet cross section $(-1/\sigma)d\sigma_7/dT$ for $y = 0.04$ and $y = 0.01$ together with the corresponding Born cross section $O(\alpha_s)$ as a function of thrust T for $\alpha_s = 0.16$. σ is defined in (7.2.4). The thrust axis is the quark momentum. The x -axis is in the direction of the second most energetic jet.

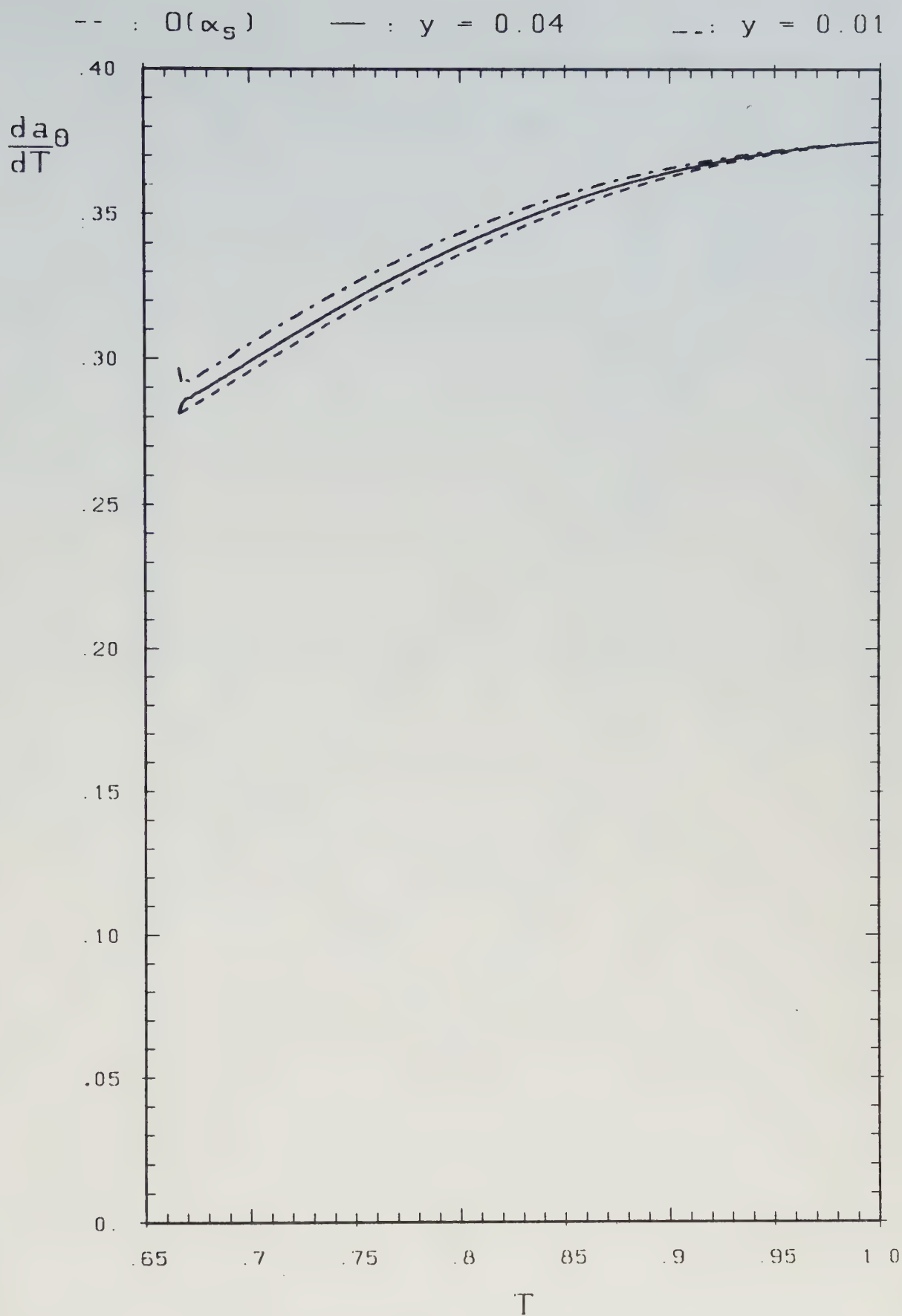


Fig.25 The $O(\alpha_s^2)$ asymmetry function da_θ/dT for $y = 0.04$ and $y = 0.01$ together with the corresponding Born term function ($O(\alpha_s)$) as a function of thrust T for $\alpha_s = 0.16$.

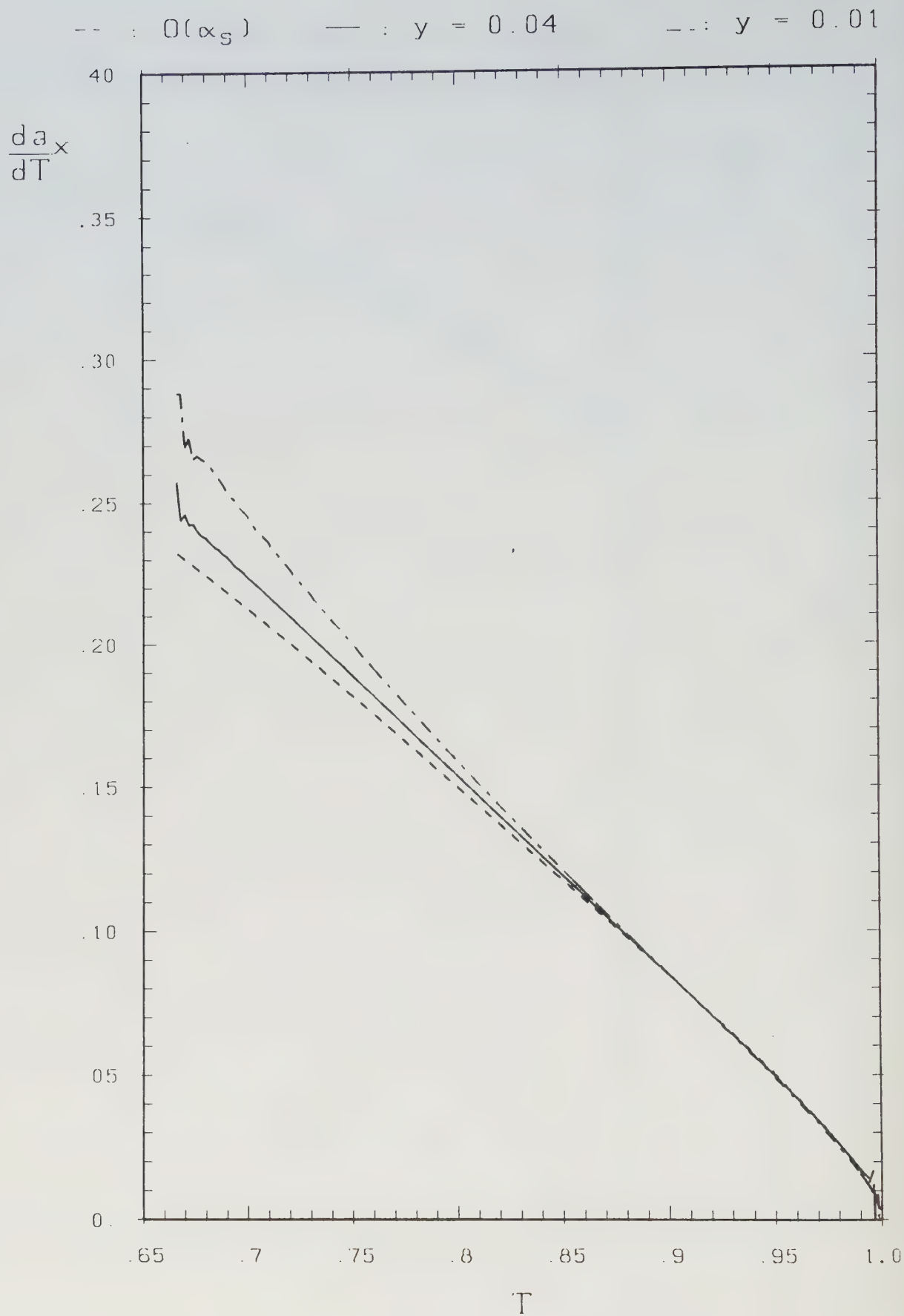


Fig.26 The $O(\alpha_s^2)$ asymmetry function da_x/dT for $y = 0.04$ and $y = 0.01$ together with the corresponding Born term function ($O(\alpha_s)$) as a function of thrust T for $\alpha_s = 0.16$.

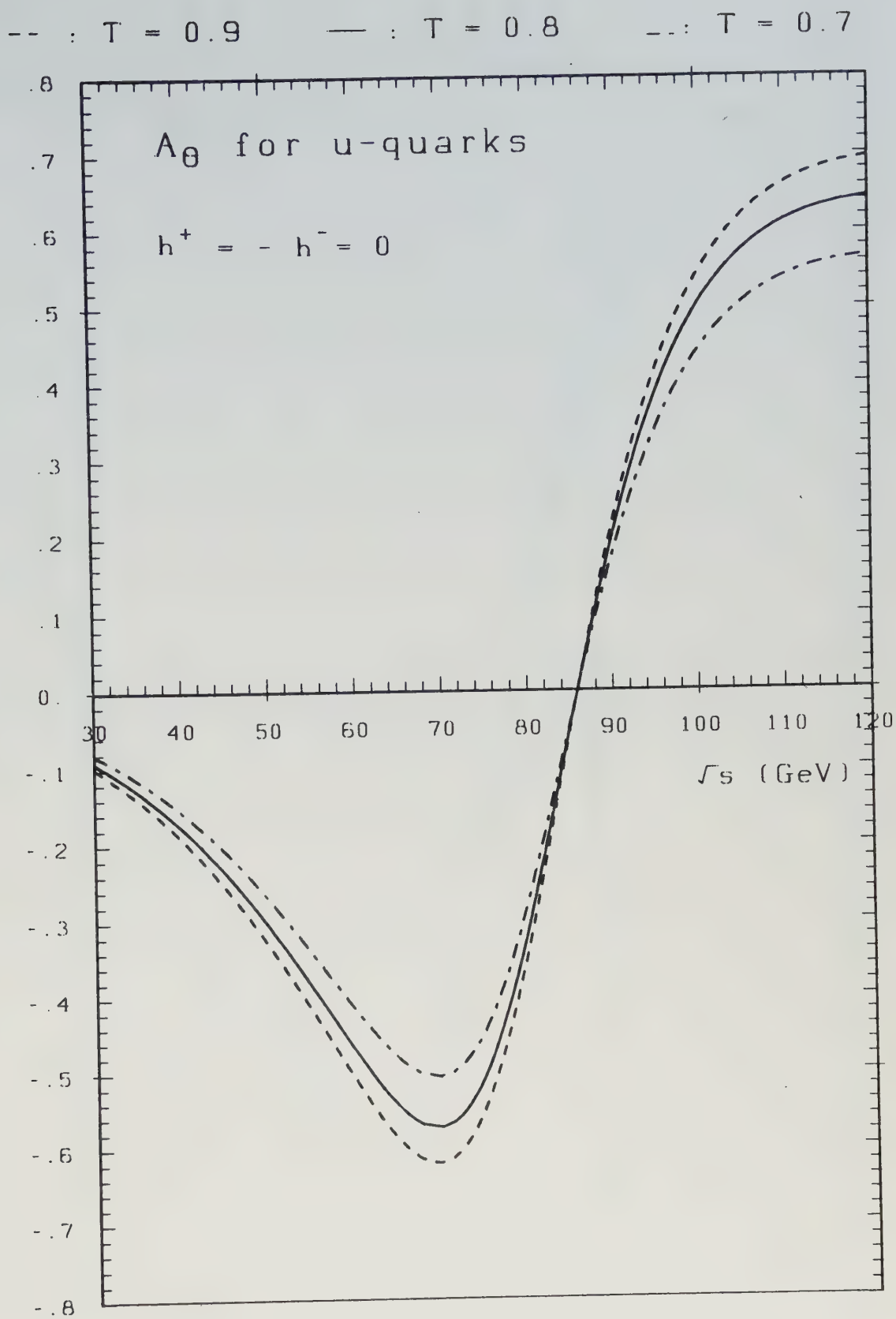


Fig.27 The $O(\alpha_s^2)$ charge asymmetry A_θ for u-type quarks as a function of the cm-energy $\sqrt{s}$ for various values of thrust T ($\alpha_s = 0.16$; $y = 0.04$) and for unpolarized beams.

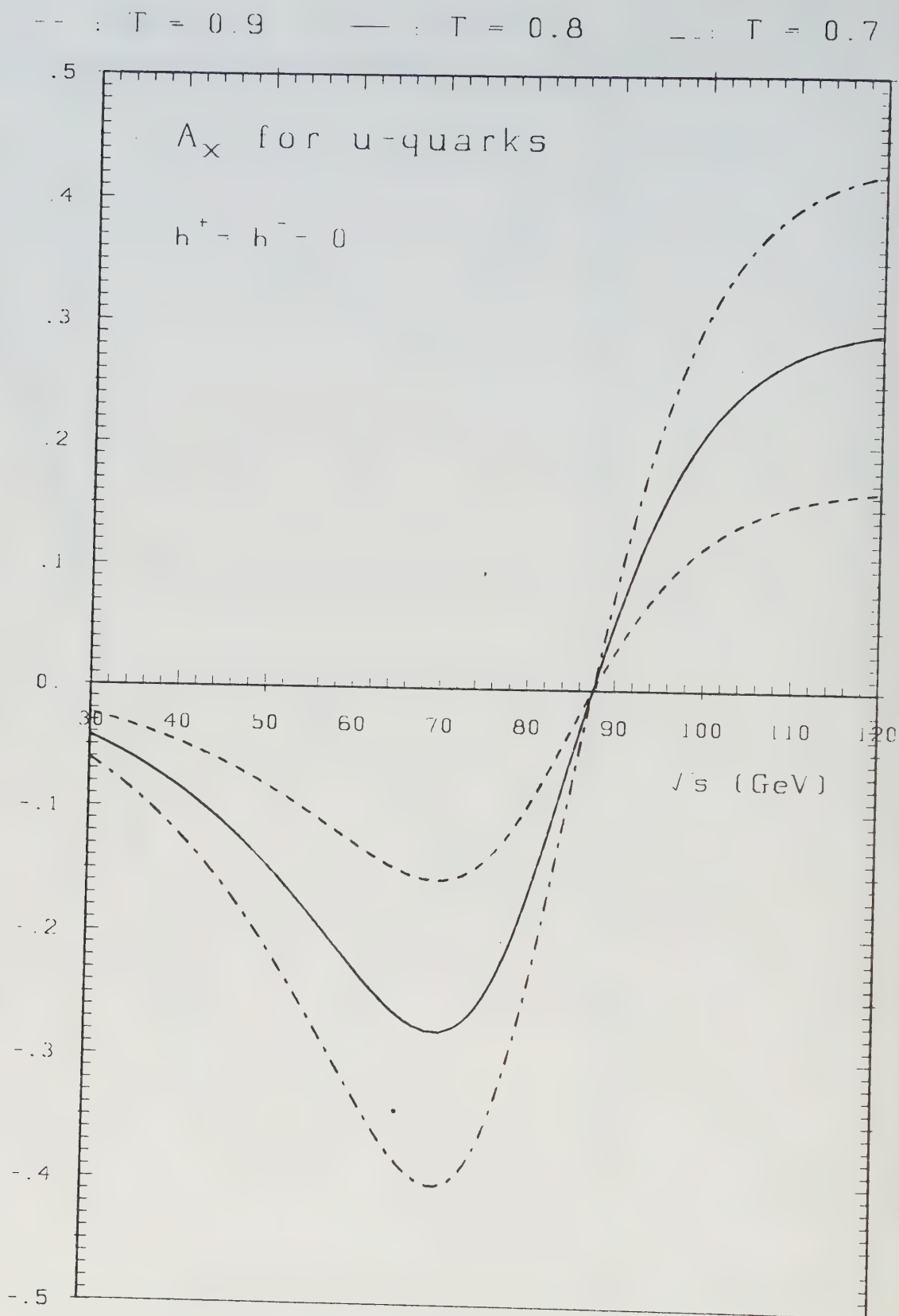


Fig.28 The $O(\alpha_s^2)$ charge asymmetry A_N for u-type quarks as a function of the cm-energy $\sqrt{s}$ for various values of thrust T ($\alpha_s = 0.16$; $y = 0.04$) and for unpolarized beams.

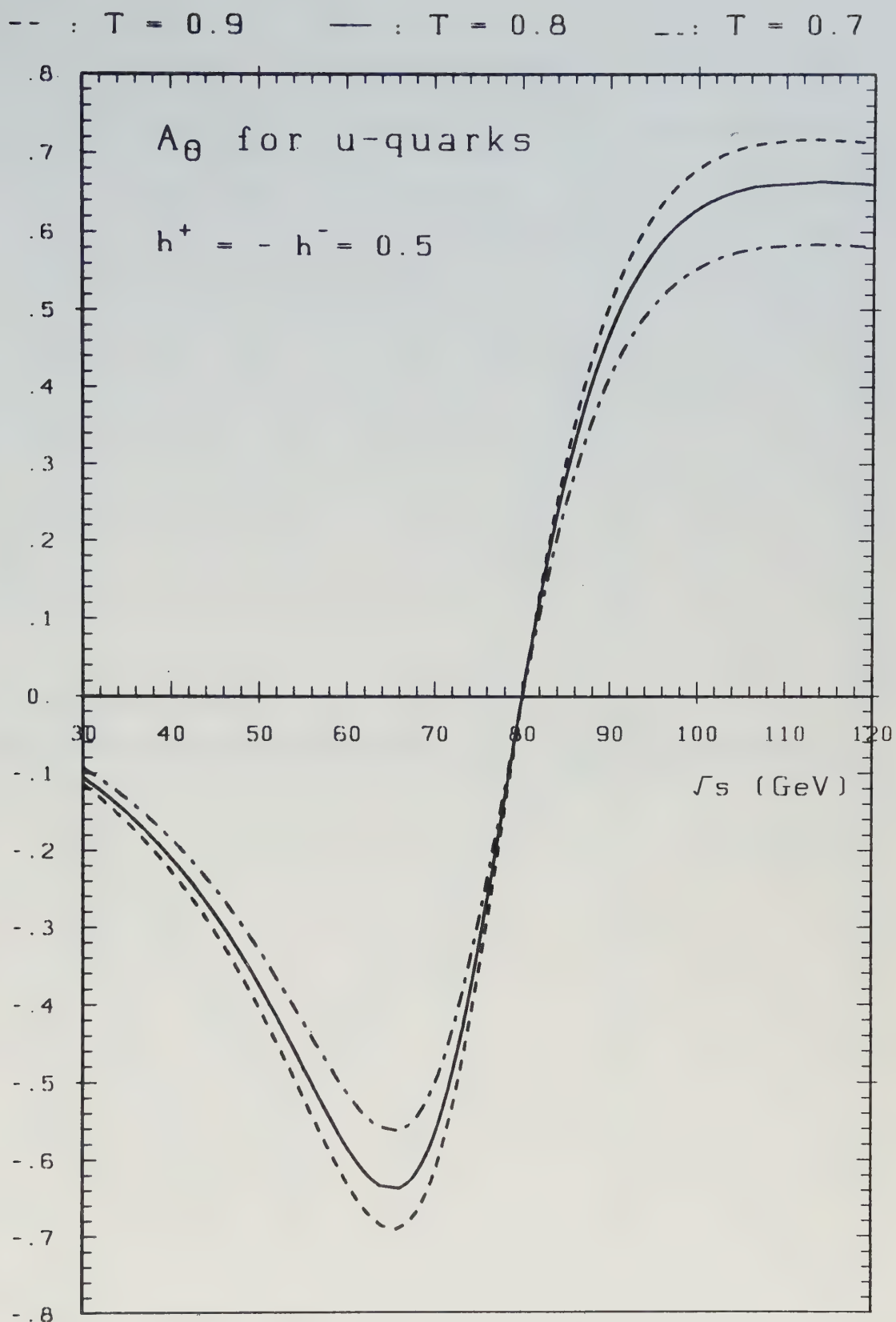


Fig.29 The $O(\alpha_s^2)$ charge asymmetry A_θ for u-type quarks as a function of the cm-energy $\sqrt{s}$ for various values of thrust T ($\alpha_s = 0.16$; $y = 0.04$) and for $h^+ = -h^- = 0.5$ where $h^\pm$ is the helicity of the electron (positron).

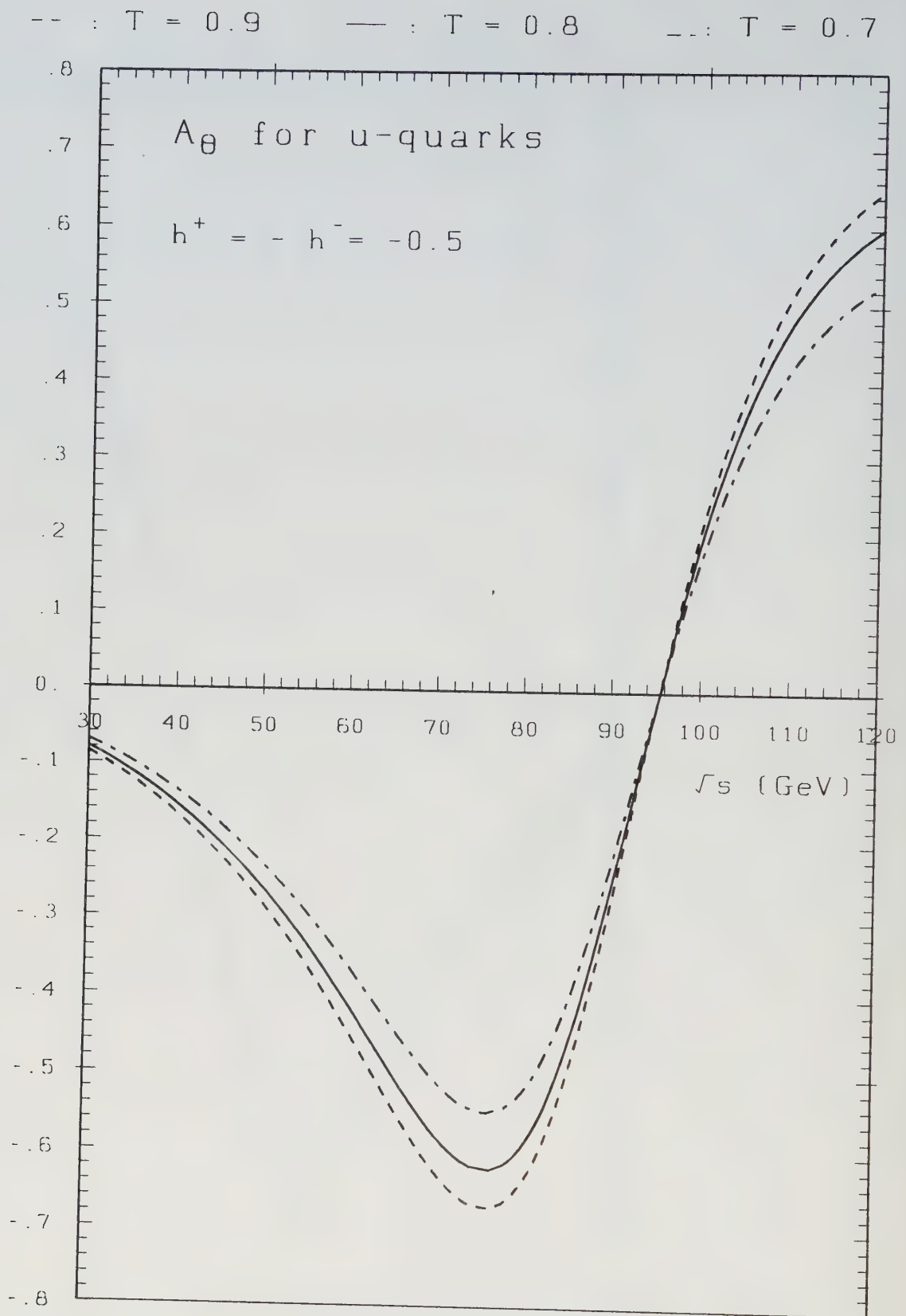


Fig.30 The $O(\alpha_s^2)$ charge asymmetry A_θ for u-type quarks as a function of the cm-energy $\sqrt{s}$ for various values of thrust T ($\alpha_s = 0.16$; $y = 0.04$) and for $h^+ = -h^- = -0.5$ where $h^\pm$ is the helicity of the electron (positron).

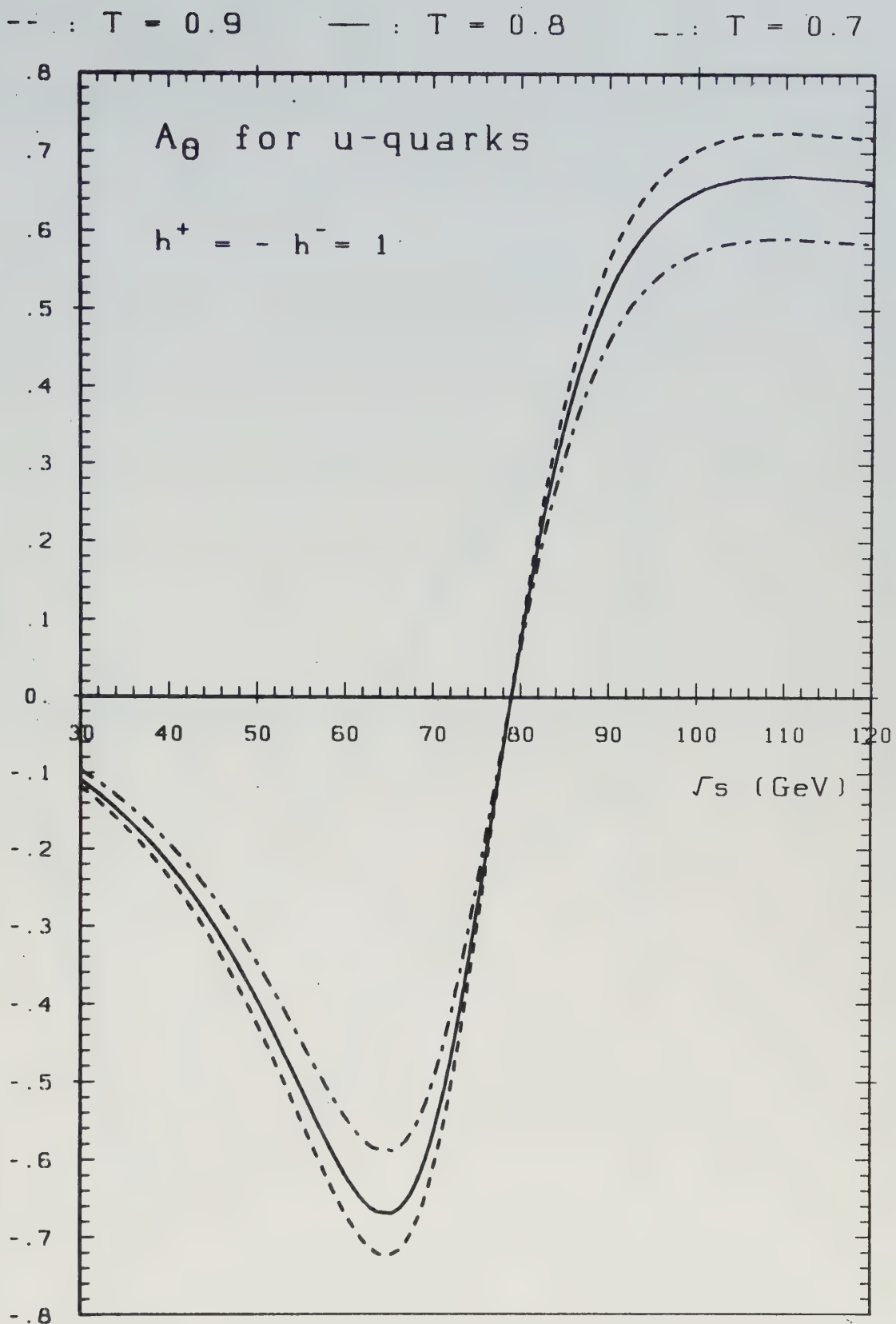


Fig.31 The $O(\alpha_s^2)$ charge asymmetry A_θ for u-type quarks as a function of the cm-energy $\sqrt{s}$ for various values of thrust T ($\alpha_s = 0.16$; $y = 0.04$) and for $h^+ = -h^- = 1.0$ where $h^\pm$ is the helicity of the electron (positron).

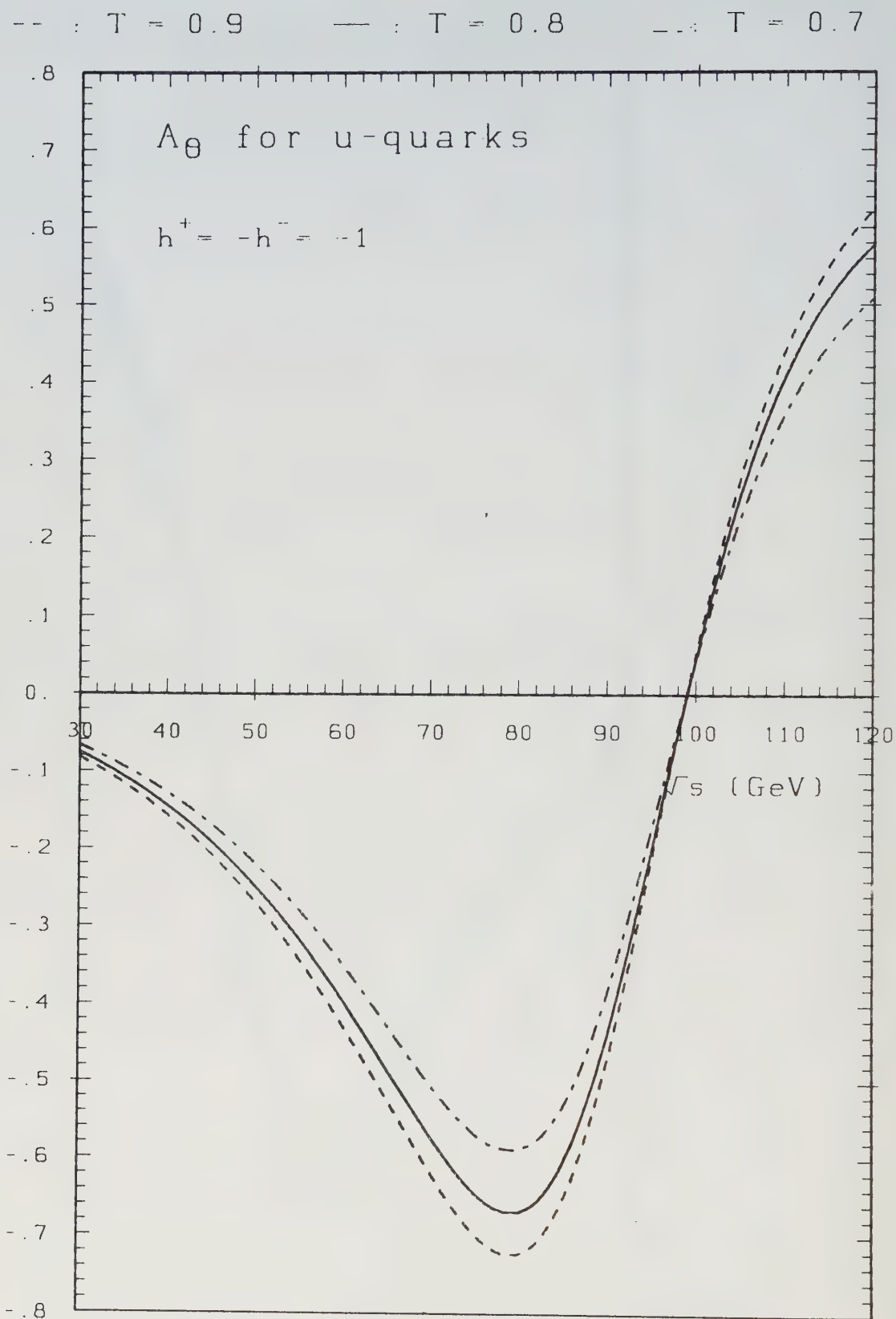


Fig.32 The $O(\alpha_s^2)$ charge asymmetry A_θ for u-type quarks as a function of the cm-energy $\sqrt{s}$ for various values of thrust T ($\alpha_s = 0.16$; $y = 0.04$) and for $h^+ = -h^- = -1.0$ where $h^\pm$ is the helicity of the electron (positron).

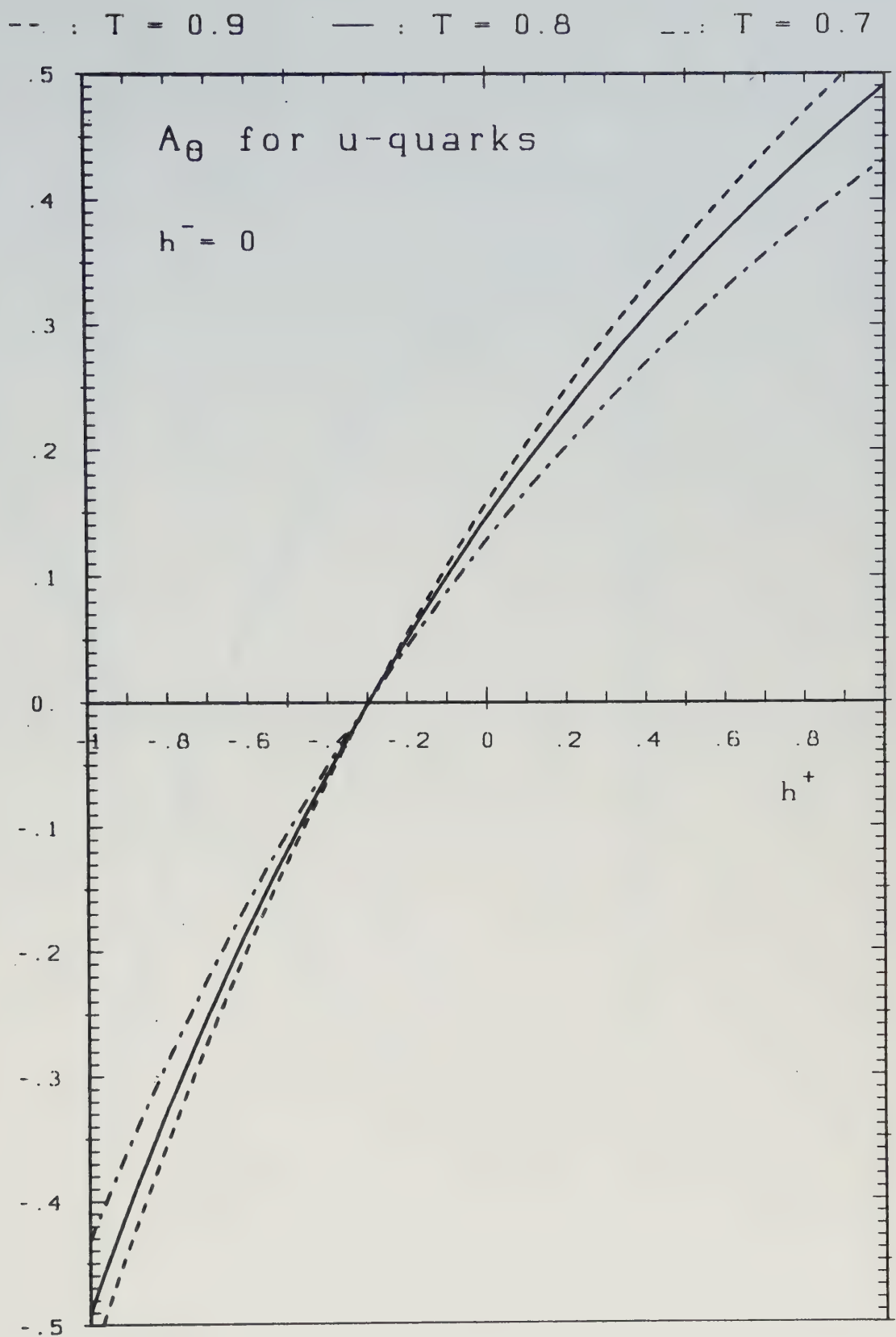


Fig.33 The $O(\alpha_s^2)$ charge asymmetry A_θ for u-type quarks on top of the Z^0 -resonance as a function of the helicity h^+ of the positron for various values of thrust T ($\alpha_s = 0.16$; $y = 0.04$) and for $h^- = 0$ where h^- is the helicity h^- of the electron

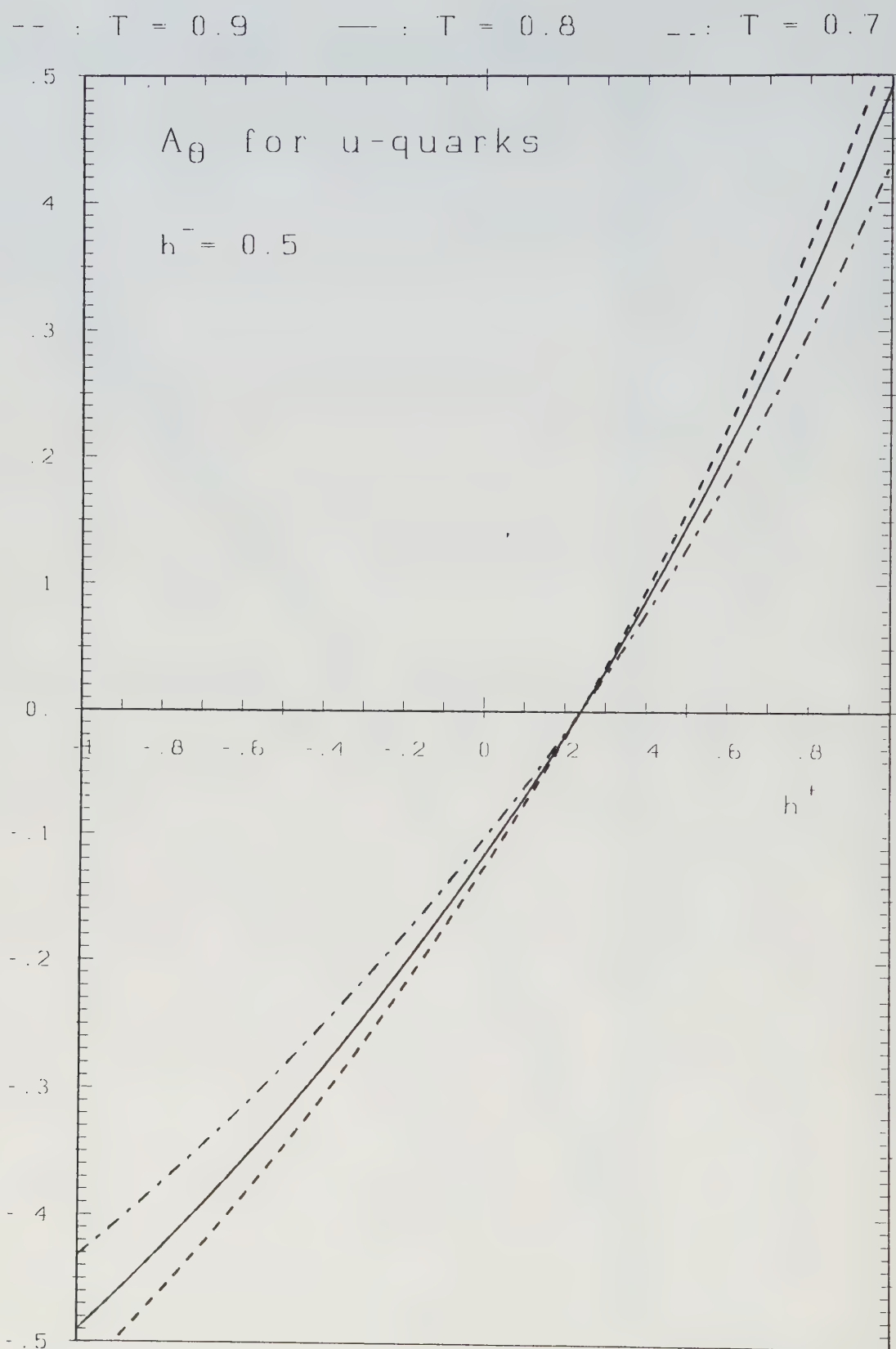


Fig.34 The $O(\alpha_s^2)$ charge asymmetry A_θ for u-type quarks on top of the Z^0 -resonance as a function of the helicity h^+ of the positron for various values of thrust T ($\alpha_s = 0.16$; $y = 0.04$) and for $h^- = 0.5$ where h^- is the helicity h^- of the electron

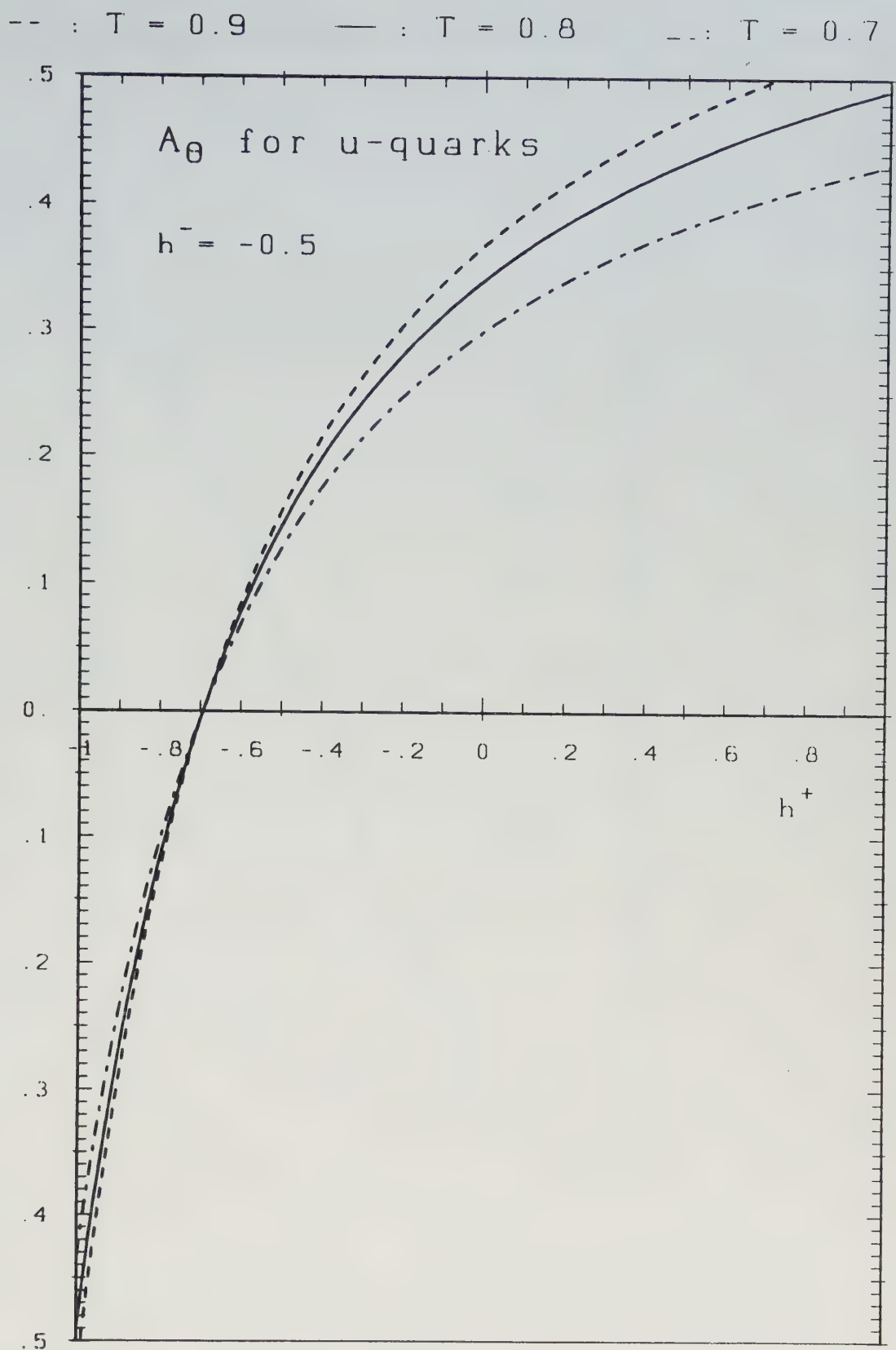


Fig.35 The $O(\alpha_s^2)$ charge asymmetry A_θ for u-type quarks on top of the Z^0 -resonance as a function of the helicity h^+ of the positron for various values of thrust T ($\alpha_s = 0.16$; $y = 0.04$) and for $h^- = -0.5$ where h^- is the helicity h^- of the electron

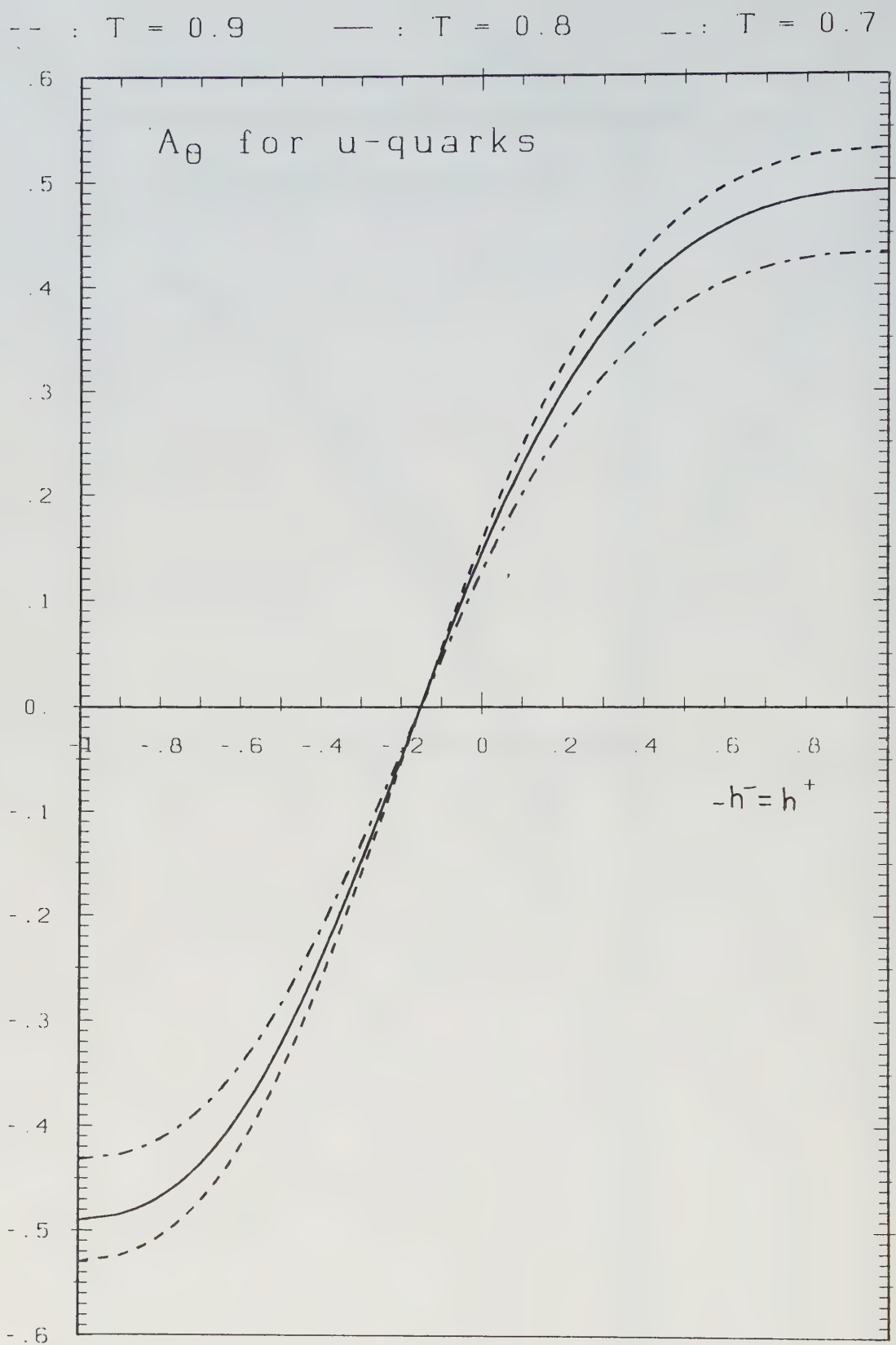


Fig.36 The $O(\alpha_s^2)$ charge asymmetry A_θ for u-type quarks on top of the Z^0 -resonance as a function of the helicity h^+ of the positron for various values of thrust T ($\alpha_s = 0.16$; $y = 0.04$) and for $h^- = -h^+$ where h^- is the helicity h^- of the electron

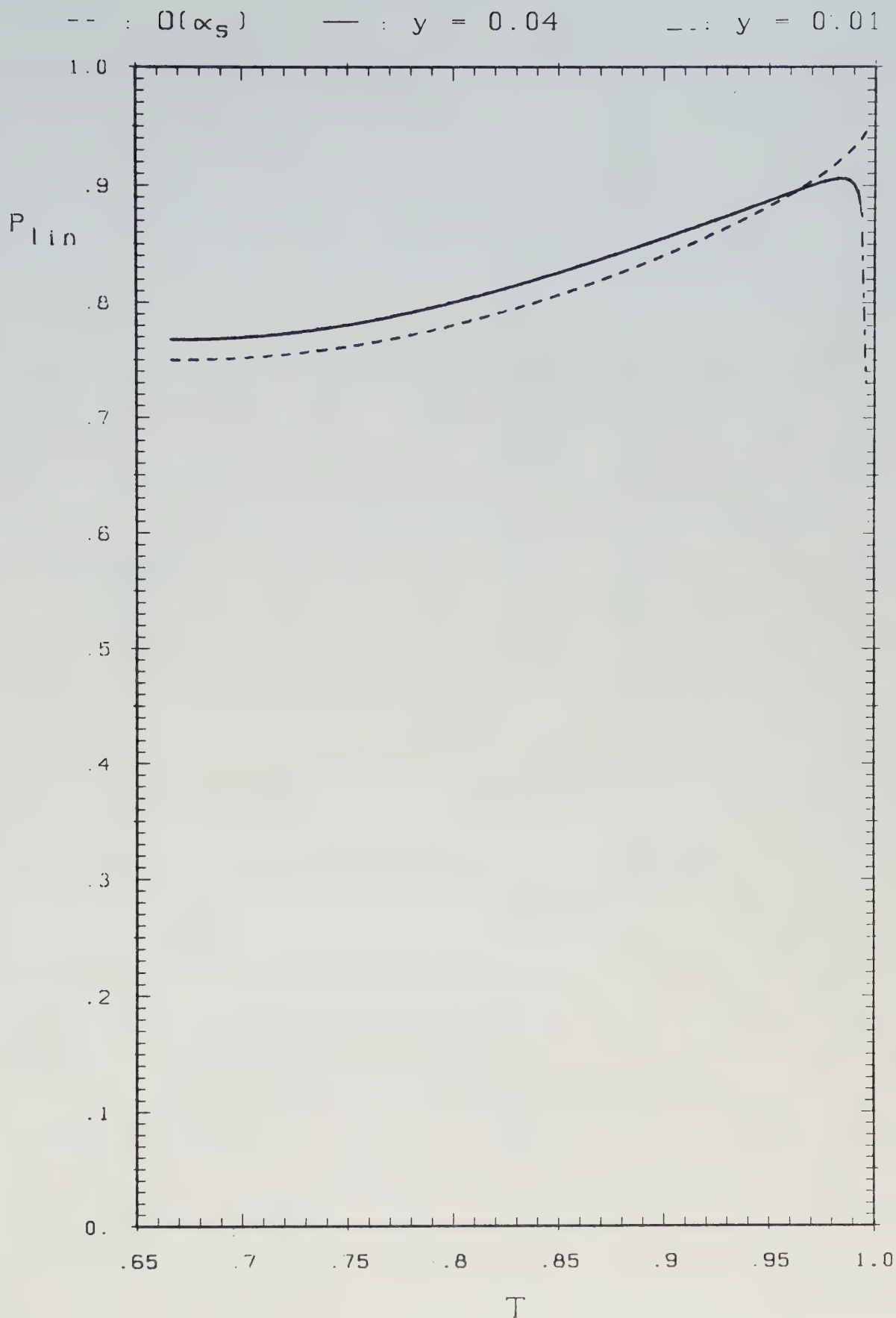


Fig.37 The $O(\alpha_s^2)$ linear polarization P_{lin} of the gluon for $y = 0.04$ and $y = 0.01$ together with the Born polarization as a function of thrust T .

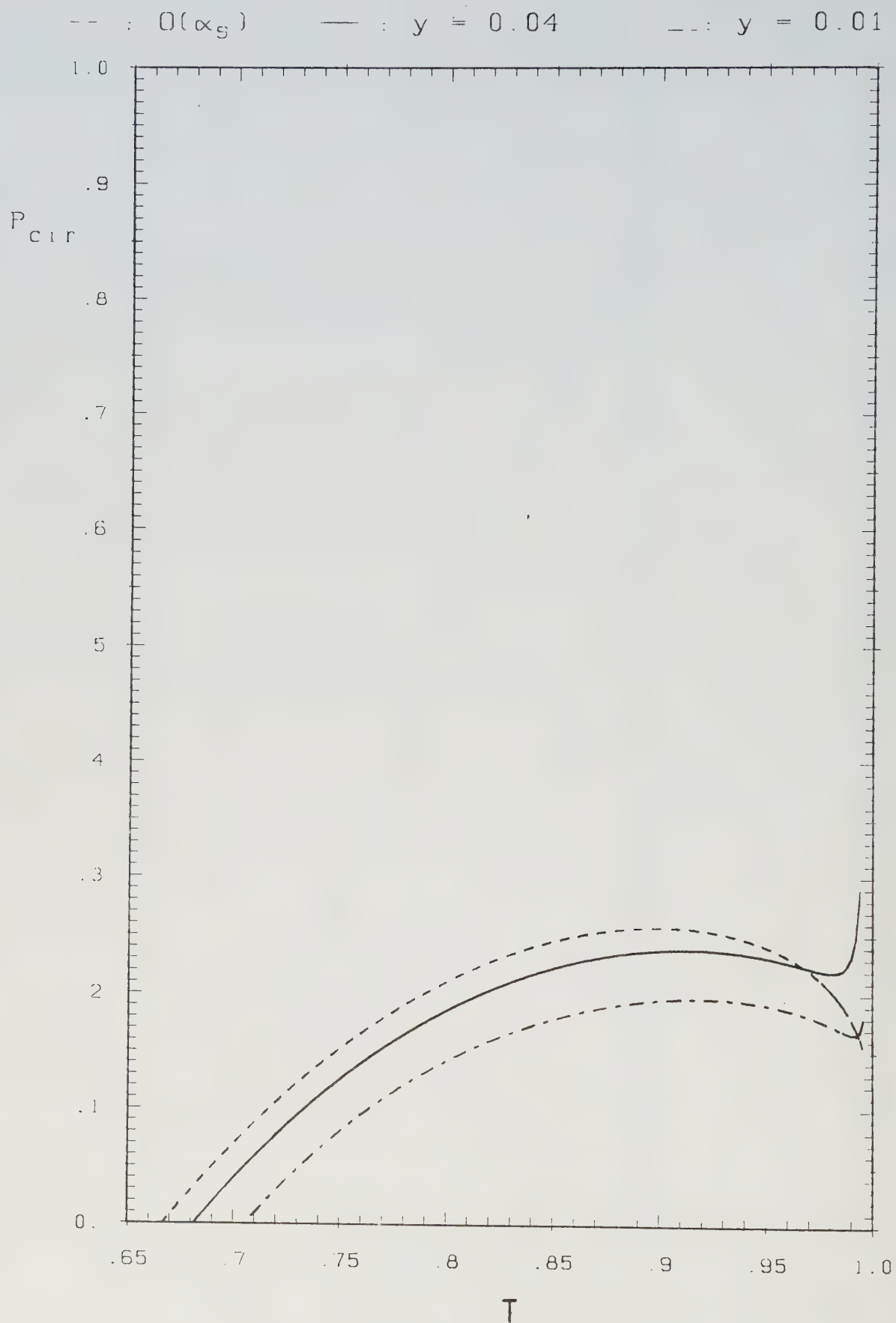


Fig.38 The $O(\alpha_s^2)$ circular polarization P_{cir} of the gluon for $y = 0.04$ and $y = 0.01$ together with the Born polarization as a function of thrust T . The thrust axis is the quark momentum.

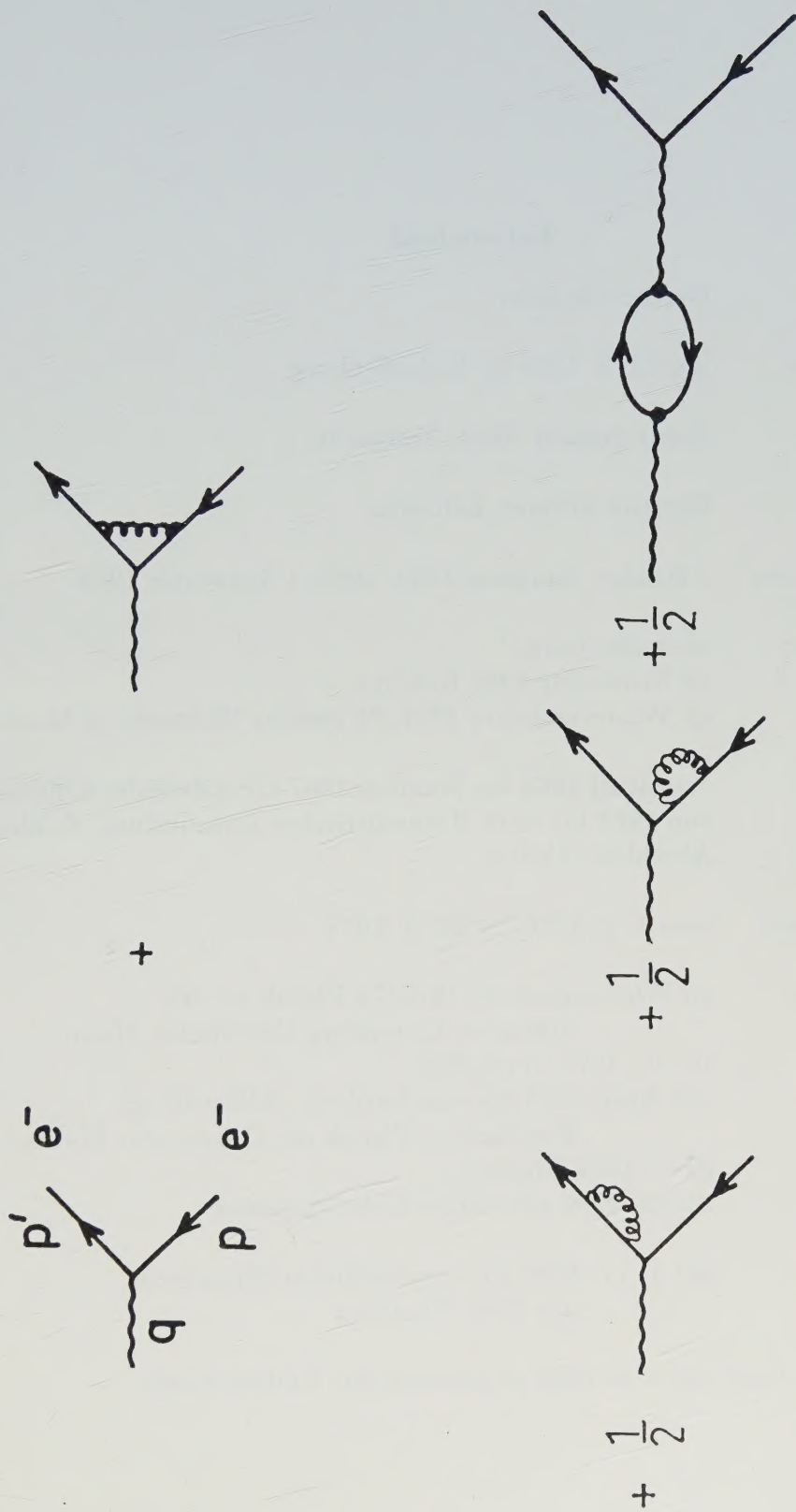


Fig.39 $\mathcal{O}(\alpha)$ Feynman diagrams to $\gamma \rightarrow e^+e^-$

Lebenslauf

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